Lecture 21: Linear Operators and Geometry

1 Linear Operators

A **linear operator** is a linear transformation where the domain is equal to the codomain.

That is it is a transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ for some n

According to what we developed in last lecture, this means that the matrix $[T]$ for T is square $(n \times n)$, so

$$T(\mathbf{v}) = [T]\mathbf{v}$$

where

$$[T] = \big(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \dots\ T(\mathbf{e}_n)\big)$$

Example: **expansion** in the x-direction. Consider the linear transformation

$$T(\mathbf{v}) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (2x, y)$

We can also have **compression** in the x-direction.

$$T(\mathbf{v}) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (1/2x, y)$

Or expansion in the y-direction

$$T(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (x, 3y)$

A **dilation** scales up equally in both directions

$$T(\mathbf{v}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (2x, 2y)$

A **contraction** scales down equally in both directions

$$T(\mathbf{v}) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (1/2x, 1/2y)$

Projection onto the x-axis

$$P_x(\mathbf{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (x, 0)$

Projection onto the y-axis

$$P_y(\mathbf{v}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{v}$$

equivalently $T(x, y) = (0, y)$

Rotation counterclockwise by an angle θ (measured in radians)

$$R_\theta(\mathbf{v}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{v}$$

or $R_\theta(x, y) = (\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y)$

Example: Rotation by $\pi/4$ counterclockwise

$R_{\pi/4} =$

Reflection through the line whose angle with the x-axis is θ

$$H_\theta(\mathbf{v}) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \mathbf{v}$$

or $H_\theta(x, y) = (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - (\cos 2\theta)y)$

Example: Reflection through the line with angle $\pi/4$.

$$H_{\pi/4} =$$

2 Successive Use of Linear Operators

If someone asks us to first scale in the x-direction by a factor of $1/2$ (contract) and then rotate by an angle of $\pi/4$ (counterclockwise), then we can write the matrices for these individual operators

To get a single linear operator for the combined operation, we multiply the operators, but be careful to get the order right.

3 Preservation of Dot Products

We say that an operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is **orthogonal**, or call it an **isometry**, or say that it **preserves dot products** if

$$T(\mathbf{u}) \cdot T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

for any $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$.

What does this means geometrically?

First of all, if $\mathbf{u} = \mathbf{v}$, it means that

Secondly, recall that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ (measured the shortest way so that $0 \leq \theta \leq \pi$), then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||}$$

On the other hand, if we call ϕ the angle between $T(\mathbf{u})$ and $T(\mathbf{v})$—again measured the shortest way so that $0 \leq \phi \leq \pi$, then

Interesting fact: to show that a linear operator T is orthogonal, it is enough to show that it preserves lengths: that $||T(\mathbf{v})|| = ||\mathbf{v}||$ for all $\mathbf{v} \in \mathbb{R}^n$. This will imply that it preserves angles (see text, Theorem 6.2.1)

4 Orthogonal Matrices

Recall that every linear transformation T has a matrix (which we write $[T]$) such that

$$T(\mathbf{v}) = [T]\mathbf{v}$$

An orthogonal operator T is a very special type of operator, so you might expect its matrix to be special too.

Indeed, the matrices for orthogonal operators are called **orthogonal matrices**, and they have special properties.

For this lecture, we're going to use A^t for the transpose of A, rather than our more usual A^T, because we don't want to get the capital T confused with transformations. You can find A^t used commonly as an alternative for A^t.

For a square matrix A, the following are equivalent:

(a) $A^t A = I$

(b) A is invertible with $A^{-1} = A^t$

(c) the columns of A form an orthonormal set of vectors

(d) the rows of A form an orthonormal set of vectors

(e) $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

(f) the linear operator T_A with $T_A(\mathbf{v}) = A\mathbf{v}$ is orthogonal

(g) $||A\mathbf{u}|| = ||A\mathbf{u}||$ for all $\mathbf{u} \in \mathbb{R}^n$

If these are true, we say that A is an **orthogonal matrix**

5 Rules for Orthogonal Matrices

(a) The transpose of an orthogonal matrix is orthogonal

(b) The inverse of an orthogonal matrix is orthogonal

(c) The product of orthogonal matrices is orthogonal

(d) If A is orthogonal, then $\det(A) = \pm 1$

(e) The only 2×2 orthogonal matrices are rotations (which have determinant equal to 1) and reflections (which have determinant equal to -1)