Lecture 20: Linear Transformations

1 Functions

Let D and E be two sets (of numbers or vectors)

A **function** $f\colon D \to E$ is a rule for associating a unique element of E to each element of D

For example, suppose $f\colon \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x^2$

With input $x = -5$, the output is

The output 25 is called the **value of** f **at** -5 or the **image of** -5 **under** f.

We also say that f **maps** -5 **to** 25

The set D is called the **domain** of the function: it is the set of allowed inputs. $f(x)$ must be defined for every x in D.

The set E is called the **codomain** of the function. It is a set where all the possible outputs reside.

Note that the domain and codomain are an important part of the definition of the function f (and not just the rule "x is taken to x^2")

We can have a function g based on the same rule $g(x) = x^2$, but with different domain and codomain $g\colon \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ is the set of all complex numbers

Then note that $f(x)$ never takes negative values

But $g(x)$ does

So $f(x)$ and $g(x)$ are different functions!

This brings up an important point: the codomain E of a function $f\colon D \to E$ may have values in it that $f(x)$ never actually has as outputs!

For example, if $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, then -3 is in the codomain $\mathbb{R}$, but

In order to specify exactly which values $f\colon D \to E$ actually has as outputs, we define the **range** of f to be the set of all outputs of f as x runs through all the values of D.

Example: $f\colon \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Find the range of f.

2 Linear Transformations

A **transformation** is a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ (where m and n are some positive integers) that takes n-dimensional vectors to m-dimensional vectors

Example: $T(x, y) = (x^2, xy, y)$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

$T(1, 2) =$

We say that a transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if both

- $T(a\mathbf{x}) = aT(\mathbf{x})$ for any scalar a and any vector $\mathbf{x}$ in the domain $\mathbb{R}^n$ [**homogeneity property**], and
- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any vectors $\mathbf{x}$ and $\mathbf{y}$ in the domain $\mathbb{R}^n$ [**additivity property**].

Example: the transformation

$$T(x, y) = (x^2, xy, y)$$

Example: the transformation

$$T(x, y) = (x + 2y, 3x - y, 5y)$$

Notice that the transformation

$$T(x, y) = (x + 2y, 3x - y, 5y)$$

can be written in matrix form as

This is not just true for the matrix $\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 5 \end{pmatrix}$.

For any $m \times n$ matrix A, the transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(\mathbf{x}) = A\mathbf{x}$ is linear

3 Algebraic Rules

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}_1$, $\mathbf{v}_2$, . . ., $\mathbf{v}_k$ and scalars a_1, a_2, . . ., a_k,

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{v}) = -T(\mathbf{v})$

(c) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

(d) $T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \ldots + a_kT(\mathbf{v}_k)$

One consequence of rule (d), is that if we know what a linear transformation does to the standard basis vectors, then we know what it does to **any vector**.

For example: suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a transformation, and we know that

$$T(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

$$T(\mathbf{e}_3) = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are the standard unit vectors.

Then if someone asks you what $T(2, 2, 1)$ is

In fact, if you want to know what $T(x, y, z)$ is for a generic vector (x, y, z), We can put this into matrix form

This is a manifestation of a general principle:

Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation: then

$$T(\mathbf{x}) = [T]\mathbf{x}$$

where $[T]$ is the matrix

$$[T] = \big(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \ldots\ T(\mathbf{e}_n)\big)$$

whose columns are given by evaluating T on the standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$ of $\mathbb{R}^n$.

4 Summary: Going Back and Forth Between Transformations and their Matrices

Now we have seen that every linear transformation is equivalent to multiplication by a matrix.

Matrix to Transformation:

If A is an $m \times n$ matrix, we used (on slide 8), the notation T_A for the transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

Transformation to Matrix:

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a transformation, then we saw on the previous slide that there is an $m \times n$ matrix $[T]$, given by

$$[T] = \big(T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \ldots\ T(\mathbf{e}_n)\big)$$

such that

$$T(\mathbf{x}) = [T]\mathbf{x}$$