Lecture 19: Dynamical Systems

1 Dynamical Systems

A **dynamical system** is a finite set of variables that change with time.

The values of the variables are recorded as the components of a vector $\mathbf{x}(t)$, called the **state** of the system at time t. For example, suppose

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

where $x_1(t)$ is the fraction of the population that does not have malaria on day t (=probability that a randomly selected person is malaria-negative), and $x_2(t)$ is the fraction of the population that has malaria on day t (=probability that a randomly selected person is malaria-positive)

Say

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x}(1) = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}, \mathbf{x}(2) = \begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix}, \ldots$$

Note that for each $t = 0, 1, 2,$ we have
$x_1(t) + x_2(t) =$

This is because the total probability should always sum to

We call a vector whose entries are nonnegative numbers whose sum is 1 a **probability vector**.

Example: is any of the following a probability vector?

$$\begin{pmatrix} 0.8 \\ 0.2 \\ 0.1 \end{pmatrix}$$

$$\begin{pmatrix} 1.1 \\ -0.2 \\ 0.1 \end{pmatrix}$$

$$\begin{pmatrix} 0.9 \\ 0.1 \\ 0.0 \end{pmatrix}$$

Suppose that we learn that on each new day

- 10 percent of the healthy people contract malaria (and 90 percent remain healthy)
- 30 percent of those infected become healthy again (and 70 percent remain infected)

Then this is a relation between $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{x}(t+1) = \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix}$.

We can put this into matrix form as

Note that each column of the matrix is a probability vector.

2 Stochastic Matrices

A **stochastic matrix** is a square matrix whose columns are all probability vectors.

Example: is any of the following a stochastic matrix?

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

3 Markov Chains

A **Markov Chain** is a dynamical system whose states are probability vectors $\mathbf{x}(t)$, and such that the state at time $t+1$ is determined from the state at time t by

$$\mathbf{x}(t+1) = A\mathbf{x}(t)$$

for some stochastic matrix A.

(The matrix A is called the **transition matrix** of the Markov chain.)

Example: our original dynamical system is a Markov chain, since

$$\mathbf{x}(t+1) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \mathbf{x}(t)$$

and the matrix is stochastic.

4 Iteration

Note that

$$\begin{aligned}
\mathbf{x}(1) &= A\mathbf{x}(0) \\
\mathbf{x}(2) &= A\mathbf{x}(1) = A(A\mathbf{x}(0)) = \\
\mathbf{x}(3) &= A\mathbf{x}(2) = A(A^2\mathbf{x}(0)) = \\
\mathbf{x}(4) &= A\mathbf{x}(3) = A(A^3\mathbf{x}(0)) =
\end{aligned}$$

More generally

$$\mathbf{x}(t) =$$

so once we know the state at time zero (that is, once we know $\mathbf{x}(0)$), we can figure out the state at all future times.

$\mathbf{x}(0)$ is called the **initial condition**

In the scenario where everyone starts out healthy on day 0, we have

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then

$$\begin{aligned}
\mathbf{x}(1) &= \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix} = \\
\mathbf{x}(2) &= \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix} = \\
\mathbf{x}(3) &= \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \\ \end{pmatrix} = \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix} \\
\mathbf{x}(4) &= \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix} = \begin{pmatrix} 0.7824 \\ 0.2176 \end{pmatrix} \\
\mathbf{x}(5) &= \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7824 \\ 0.2176 \end{pmatrix} = \begin{pmatrix} 0.76944 \\ 0.23056 \end{pmatrix} \\
&\vdots \\
\mathbf{x}(20) &= \begin{pmatrix} 0.7500091403961001574 4 \\ 0.2499908596038998425 6 \end{pmatrix} \\
&\vdots
\end{aligned}$$

This seems to be tending to a limit.

What if we started off with a different initial condition, say half of the people are infected on day 0.

$$\mathbf{x}(0) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$\mathbf{x}(1) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \qquad \\ \qquad \end{pmatrix} =$$

$$\mathbf{x}(2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} \qquad \\ \qquad \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$$

$$\mathbf{x}(3) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix} = \begin{pmatrix} 0.696 \\ 0.304 \end{pmatrix}$$

$$\mathbf{x}(4) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.696 \\ 0.304 \end{pmatrix} = \begin{pmatrix} 0.7176 \\ 0.2824 \end{pmatrix}$$

$$\mathbf{x}(5) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7176 \\ 0.2824 \end{pmatrix} = \begin{pmatrix} 0.73056 \\ 0.26944 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{x}(20) = \begin{pmatrix} 0.74999085960389984256 \\ 25000914039610015744 \end{pmatrix}$$

$$\vdots$$

On the other hand, suppose we had another Markov chain with

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

with transition matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

so that

$$\mathbf{x}(t+1) = A\mathbf{x}(t),$$

and suppose that the initial condition was

$$\mathbf{x}(0) = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}$$

Then

$$\mathbf{x}(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}(0)$$

$$\mathbf{x}(0) = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}$$

$$\mathbf{x}(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix}$$

$$\mathbf{x}(2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix}$$

$$\mathbf{x}(3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix}$$

$$\mathbf{x}(4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \qquad \\ \\ \\ \end{pmatrix}$$

$$\vdots$$

So the state is not tending to a limit: it just cycles between three different states.

5 Classifying Markov Chains

Here is a way to guarantee that the state of the Markov chain $\mathbf{x}(t+1) = A\mathbf{x}(t)$ approaches some limit.

Look at the powers of the transition matrix A, and see if you can find a power with **all entries** nonzero. For example, for our original Markov chain, we had

$$A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$$

which already has all entries nonzero.

On the other hand, if you look at our second Markov chain, with

$$A^1 = A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$A^3 = AA^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} =$$

$$A^4 = AA^3 = AI = A$$

$$A^5 = A^2A^3 = A^2I = A^2$$

$$A^6 = A^3A^3 = II = I$$

$$\vdots$$

every single power has a zero entry.

For a third example, suppose that

$$A = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix}$$

Note that $A^1 = A$ has a zero entry, but

$$A^2 = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix} =$$

and A^2 has no zero entries.

6 Regular Stochastic Matrices and Markov Chains

A stochastic matrix A is said to be **regular**, if there is some positive k such that A^k has no zero entries.

Example: $A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$ is

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is

$A = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix}$ is

A Markov chain $\mathbf{x}(t+1) = A\mathbf{x}(t)$ is said to be **regular** if its transition matrix A is a regular stochastic matrix.

So the Markov chains $\mathbf{x}(t+1) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$ and $\mathbf{x}(t+1) = \begin{pmatrix} 0 & 1/2 \\ 1 & 1/2 \end{pmatrix}$ are

And the Markov chain $\mathbf{x}(t+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is

Theorem: If $\mathbf{x}(t+1) = A\mathbf{x}(t)$ is a regular Markov chain and $\mathbf{x}(0)$ is **any** initial condition, then $\mathbf{x}(t) = A^t\mathbf{x}(0)$ will tend to a limit (which does not depend on the particular choice of $\mathbf{x}(0)$) as $t \to \infty$.

We saw this on slides 9 and 10.

The theorem tells us that there is a limit, but what is that limit?

It should be a vector $\mathbf{x}$ that does not change when we keep multiplying it with A (on the left), that is

That is, it should be a

But remember that it also has to be a probability vector

For instance, with out Markov chain $\mathbf{x}(t+1) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix},$$

we look for fixed points of A, so we solve

So the fixed points of $A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$ are vectors of the form $t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Remember that we need a probability vector, so we must choose t accordingly.

So for the Markov chain $\mathbf{x}(t+1) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix},$$

no matter what the initial state $\mathbf{x}(0)$, we will have $\mathbf{x}(t) = A^t\mathbf{x}(0)$ tending toward $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$ as $t \to \infty$.

We found that there was exactly one fixed point of A that was a probability vector.

Could there be more than one in other cases?

Theorem: If A is a **regular** stochastic matrix, then A has exactly one fixed point $\mathbf{x}$ that is also a probability vector.