Lecture 18: Eigenvalues and Eigenvectors, Continued

1 Complex Eigenvalues

Consider the matrix $A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$

We can calculate its characteristic polynomial

Which has roots

2 Factoring Polynomials

The Fundamental Theorem of Algebra: If

$$p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0$$

is a polynomial whose coefficients p_0, p_1, …, p_{n-1} are numbers (real, imaginary, or complex), then we can factor $p(\lambda)$ completely:

$$p(\lambda) = (\lambda - r_1)(\lambda - r_2)\cdots(\lambda - r_{n-1})(\lambda - r_n)$$

where r_1, r_2, …, r_n are numbers (possibly complex, even if the original coefficients p_0, p_1, …, p_{n-1} were real).

For example, from the previous slide

The numbers $r_1, r_2, \ldots, r_n$ are called the **roots** of the polynomial $p(\lambda)$.

A number might appear more than once on this list.

For example

$$x^3 - 6x^2 + 9x - 4 = (x-1)(x-1)(x-4)$$

so the roots of this polynomial are 1 and 4.

The root 4 is called a

The root 1 is called a

The **multiplicity** of a root is the number of times it appears in the factorization.

If you add up the multiplicities of all the roots, you should get the degree of the polynomial

3 Finding Integer Roots

We can always find the roots of a quadratic polynomial with the quadratic formula, but for higher degree polynomials, it can be difficult.

Sometimes integer roots can be found by trial and error.

To do this, we make use of the following fact:

If $p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0$ is a polynomial, and the coefficients $p_0, p_1, \ldots, p_{n-1}$ are all integers, then $p(\lambda)$ does not need to have integer roots, but if it does have any, they must be divisors of p_0.

By a "divisor" of p_0, we mean an integer n (which can be positive or negative), so that p_0/n is also an integer.

So, for instance, the divisors of 6 are

Example: Let's see if $p(\lambda) = \lambda^3 - 5\lambda^2 + 13\lambda - 21$ has any integer roots.

4 Algebraic Multiplicity

The **algebraic multiplicity** of an eigenvalue of a square matrix A is its multiplicity as a root of the characteristic polynomial $\det(\lambda I - A)$.

Example: Find the eigenvalues of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$ and for each one, state its algebraic multiplicity.

5 Eigenvalues of Diagonal and Triangular Matrices

We saw on the previous slide that the eigenvalues of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$, if we list them with their algebraic multiplicities are 1, 1, and 4

More generally

6 Eigenvalues, Determinant, and Trace

For our upper triangular matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix}$, the eigenvalues (listed with multiplicity) are 1, 1, and 4.

Note that $\det(A) =$

and $\mathrm{tr}(A) =$

So for upper triangular matrices, the determinant is

and the trace is

The same principle holds for lower triangular and diagonal matrices.

7 2×2 Matrices

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix, then

$\det(A) =$

$\mathrm{tr}(A) =$

And the characteristic polynomial, $\det(\lambda I - A) =$

For example, if $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$

$\det(A) =$

$\mathrm{tr}(A) =$

$\det(\lambda I - A) =$

In fact, For **any** square matrix A,

$\det(A)$ is

$\operatorname{tr}(A)$ is

8 Eigenspaces of Real Symmetric 2×2 Matrices

Note that our matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ is real and symmetric

(By "real", we just mean that all its entries are real numbers)

We have calculated that its eigenvalues are $\lambda = 3$ and -7

Recall from Lecture Notes 17 that the eigenspace for $\lambda = 3$ is the span of the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, that is the line with vector equation

And the eigenspace for $\lambda = -7$ is span $\left(\begin{pmatrix} 1 \\ -3 \end{pmatrix} \right)$ which is the line

We can plot these two lines

Note that they are perpendicular because

This is a manifestation of a general principle. There are two kinds of real symmetric 2×2 matrices:

(1). Diagonal real symmetric matrices with identical diagonal entries: $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(2). All other real symmetric matrices: either non-diagonal $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $b \neq 0$, or diagonal but with different entries on the diagonal $C = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ with $a \neq c$.

It turns out that matrices of types B and C always have two distinct real eigenvalues

Each of these eigenvalues has an eigenspace which is one-dimensional (a line through the origin)

And these two lines are perpendicular to each other.