Lecture 17: Eigenvalues and Eigenvectors

1 Null Space of a Matrix

If A is an matrix, the **null space of** A is

We write $\mathrm{null}(A)$ to mean the null space of A

Example: If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 3 & 6 & 5 \end{pmatrix}$, then $\mathrm{null}(A)$ is

2 Fixed Points

If A is an $n \times n$ matrix, then a **fixed point** of A is a vector $\mathbf{x}$ such that

Of course $A\mathbf{0} = \mathbf{0}$, so $\mathbf{0}$ is always a fixed point. We call it the **trivial fixed point**.

To find the fixed points of A, just solve

That is, just find

Example: The fixed points of $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ are

3 A Unifying Theme

The null space of a square matrix A is the set of solutions of

$$A\mathbf{x} = \mathbf{0}$$

and then the set of fixed points A is the set of solutions of

$$A\mathbf{x} = \mathbf{x}$$

These are two special cases of a general phenomenon: they are vectors $\mathbf{x}$ for which

where λ is some scalar.

Setting $\lambda = 0$ gives

Setting $\lambda = 1$ gives

4 Eigenvalues and Eigenvectors

If A is a square matrix, then a scalar λ is a called an **eigenvalue of** A if there is some vector $\mathbf{x} \neq \mathbf{0}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

If λ is an eigenvalue of a square matrix A, then any nonzero vector $\mathbf{x}$ that satisfies

$$A\mathbf{x} = \lambda\mathbf{x}$$

is called an **eigenvector of A corresponding to eigenvalue** λ.

Example: Is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$?

Is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$?

5 Finding Eigenvalues

Recall that an eigenvalue λ of a square matrix A is a scalar such that

has a **nontrivial** solution.

This will happen if and only if $\lambda I - A$ is

Example: Find the eigenvalues of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

6 Characteristic Polynomial

For an $n \times n$ matrix A, the polynomial of degree n in λ given by

$$p(\lambda) = \det(\lambda I - A)$$

is called the **characteristic polynomial of** A.

So, for example, we found the characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ to be

The eigenvalues of A are the roots of the characteristic polynomial.

7 Finding Eigenvectors

Find all the eigenvectors of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ corresponding to eigenvalue 3.

Find all the eigenvectors of $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$ corresponding to eigenvalue -7.

8 Eigenspaces

If λ is an eigenvalue of the square matrix A, then the eigenvectors of A associated with λ are the $\mathbf{x} \neq \mathbf{0}$ with

That is they are the **nonzero** elements of

If λ is an eigenvalue of A, we define **eigenspace** of A associated with λ to be $\text{null}(\lambda I - A)$

* It is a subspace because it the solution set of a homogeneous equation

* It consists of the eigenvectors associated with λ along with the zero vector (subspaces always contain $\mathbf{0}$)

9 Summary: Finding Eigenvalues and Eigenspaces

To find the eigenvectors of a square

1. Find the characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ of A
2. Find the roots of this polynomial: these are the eigenvalues
3. For each eigenvalue λ, the eigenspace corresponding to it is $\text{null}(\lambda I - A)$: this is the set of all eigenvectors for λ plus the zero vector

10 Eigenvalues/Eigenvectors of a Power of a Matrix

Suppose that A is a square matrix with eigenvalue λ

And suppose $\mathbf{x}$ is an eigenvector of A corresponding to λ

If k is a positive integer, then

$A^k\mathbf{x} =$

Example: Find $\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}^4 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Recall that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix with eigenvalue 3.

11 Unifying Theorem

If A is an $n \times n$ matrix, then the following statements are equivalent

(a). The reduced row echelon form of A is I_n.

(b). A is expressible as a product of elementary matrices.

(c). A is invertible.

(d). $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e). $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.

(f). $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b} \in \mathbb{R}^n$.

(g). The columns of A are linearly independent.

(h). The rows of A are linearly independent.

(i). $\det(A) \neq 0$.

(j). $\lambda = 0$ is not an eigenvalue of A.

We have proved the equivalence of (a)–(i) already.

To see that (j) is equivalent to (i)