Lecture 16: The Geometry of Complex Numbers

1 The Complex Plane

We can imagine the complex number $a + bi$ as the point (a, b) in two-dimensional space

Example: $1 + 2i$ and $3 - i$ in the plane

2 Addition

Example: $(1+2i)+(3-i)$

3 Subtraction

Example: $(1 + 2i) - (3 - i)$

4 Conjugation

Example $\overline{1+2i}$

5 Polar Form

We can measure the angle ϕ from positive x-axis to our vector, counting counterclockwise rotation as positive.

This angle is called the **argument**

The argument is not unique because, for instance 0 and 2π give the same direction.

The **principal argument** is the unique angle ϕ with $-\pi < \phi \leq \pi$.

Example: $5 - 5i$

6 Multiplication

If we multiply

$$z_1 = |z_1|(\cos\phi_1 + i\sin\phi_1)$$
$$z_2 = |z_2|(\cos\phi_2 + i\sin\phi_2)$$

we get

Geometrically, this means that $z_1 z_2$ has length equal to $|z_1||z_2|$

and angle equal to $\phi_1 + \phi_2$

7 Inversion

If $z = |z|(\cos\phi + i\sin\phi)$, then note that

$$z^{-1} = \frac{\overline{z}}{|z|^2} =$$

Geometrically, this means that z^{-1} has length equal to $1/|z|$

and angle equal to $-\phi$

8 Division

If we divide z_1 by z_2 where

$$z_1 = |z_1|(\cos\phi_1 + i\sin\phi_1)$$
$$z_2 = |z_2|(\cos\phi_2 + i\sin\phi_2)$$

this is the same as $z_1 z_2^{-1}$

Geometrically, this means that z_1/z_2 has length equal to $|z_1|/|z_2|$

and angle equal to $\phi_1 - \phi_2$

9 DeMoivre's Formula

If $z = |z|(\cos\phi + i\sin\phi)$, then

z^n is what you get by multiplying n copies of this together

Remember that lengths multiply, so the length of z^n is just

$$\underbrace{|z| \cdot |z| \cdot \cdots \cdot |z|}_{\text{n } copies}$$

When you multiply complex numbers, the angles add, so the angle is

$$\underbrace{\phi + \phi + \cdots + \phi}_{\text{n } copies}$$

So

$$z^n = |z|^n(\cos(n\phi) + i\sin(n\phi))$$

If $|z| = 1$, then $z = \cos\phi + i\sin\phi$, and so

Example: If $z = 5 - 5i$, compute z^4.

10 Euler's Formula

If θ is real, we have

$$e^{i\theta} =$$

which is Euler's Formula.
If $z = a + bi$ is complex, then we have

$$e^{z} = e^{a+bi} =$$

Example: $e^{-3-2i} =$

The basic rules of exponents still work. If u and v are complex numbers

1. $e^{u+v} = e^u e^v$
2. $e^{-u} = \frac{1}{e^u}$
3. $e^{u-v} = \frac{e^u}{e^v}$