Lecture 15: The Algebra of Complex Numbers

1 Motivation

The quadratic equation

$$x^2 + 1 = 0$$

has no solution in $\mathbb{R}$, because the square of a real number is nonnegative.

In the Renaissance, scholars found it useful to formally introduce new numbers that solve such equations

We therefore introduce a new number i with the property that

$$i^2 = -1$$

Then i solves the above equation, as does $-i$.

2 Complex Numbers

Along with i, we introduce all the numbers $a+bi$ (with a and b real).

All the numbers of this form are called **complex numbers**.

Two special kinds of complex number

* If $b = 0$, then

* If $a = 0$, then

3 Real and Imaginary Parts

If $a + bi$ is a complex number, then

* The **real part** of $a + bi$ is a, written $\mathrm{Re}(a + bi) = a$

* The **imaginary part** of $a + bi$ is b, written $\mathrm{Im}(a + bi) = b$.

NOTE that, the way we defined it, the imaginary part is a real number!

$\mathrm{Re}(3 - 4i) =$

$\mathrm{Im}(3 - 4i) =$

4 Equality of Complex Numbers

We have $a+bi = c+di$ if and only if $a = c$ and $b = d$

$3+4i \neq 3+3i$

5 Arithmetical Operations for Complex Numbers

Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

$(2+3i)+(1-5i) =$

Subtraction: $(a+bi)-(c+di) = (a-c)+(b-d)i$

$(2+3i)-(1-5i) =$

Negation: $-(a+bi) = -a-bi$

$-(1-5i) =$

Multiplication of Complex Numbers

$$(a + bi)(c + di) =$$

$$(2 + 3i)(1 - 5i) =$$

Division of Complex Numbers

$(a + bi)/(c + di) =$

$(2 + 3i)/(1 - 5i) =$

Conjugation: the conjugate of $a + bi$ is

$$\overline{a + bi} = a - bi$$

$\overline{1 - 5i} =$

Absolute Value (Modulus): the modulus of $a + bi$ is

$$|a + bi| = \sqrt{a^2 + b^2}$$

$|1 - 5i| =$

Then it turns out that if $z = a + bi$, then

$z\overline{z} =$

$(1 - 5i)(\overline{1 - 5i}) =$

Multiplicative inverse: the multiplicative inverse of z is

$1/(1 - 5i) =$

6 Algebraic Rules for Conjugation

If z_1 and z_2 are complex numbers, then

(a). $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(b). $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(c). $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(d). $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$

(e). $\overline{\overline{z_1}} = z_1$

7 Algebraic Rules for the Absolute Value

If z_1 and z_2 are complex numbers, then

(a). $|\overline{z_1}| = |z_1|$

(b). $|z_1 z_2| = |z_1||z_2|$

(c). $|z_1/z_2| = |z_1|/|z_2|$

(d). $|z_1 + z_2| \leq |z_1| + |z_2|$

(e). $|z_1 - z_2| \leq |z_1| - |z_2|$

8 Solving Quadratic Equations

If we have the quadratic

$$ax^2 + bx + c = 0$$

Then the solutions are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, the equation

$$2x^2 - 5x + 4 = 0$$

is satisfied for