Lecture 14: Cross Products, Areas, and Volumes

1 Definition of Cross Product

The **cross product** is an operation that takes
as inputs: two vectors in $\mathbb{R}^3$
as output: a vector in $\mathbb{R}^3$ as its output.

It is **only** defined for $\mathbb{R}^3$

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\ &= \end{aligned}$$

An easy way to remember this is

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

Example: $(1, 2, -3) \times (4, 0, 1) =$

2 Algebraic Rules for Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, and a is a scalar, then

(a). $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(b). $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

(c). $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$

(d). $a(\mathbf{u} \times \mathbf{v}) = (a\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (a\mathbf{v})$

(e). $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f). $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

3 Geometric Interpretation

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$

with angle θ between them

then $\mathbf{u} \times \mathbf{v}$ is the vector in $\mathbb{R}^3$

* whose length is

* which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$

* with the sign given by the right hand rule

4 Geometric Consequences

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then

(a). $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(b). $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

Also $||\mathbf{u} \times \mathbf{v}||$ is the area of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides

Example: Determine the area of the parallelogram in $\mathbb{R}^3$ whose sides are given by $\mathbf{u} = (1, 2, -3)$ and $\mathbf{v} = (4, 0, 1)$

5 2×2 Determinants and Area

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are in $\mathbb{R}^2$,

the area of the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is

Example: Determine the area of the parallelogram in $\mathbb{R}^2$ whose sides are given by $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (2, 3)$.

6 Parallelepipeds

A **parallelepiped** in $\mathbb{R}^3$ is the 3-dimensional generalization of a parallelogram

7 Volume of a parallelepiped

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ determine a parallelepiped in $\mathbb{R}^3$.

Then the volume of the parallelepiped is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

(see book for proof)

Now since

$$\begin{aligned}
\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\
\mathbf{v} \times \mathbf{w} &= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k},
\end{aligned}$$

we have

Example: Determine the area of the parallelepiped in $\mathbb{R}^3$ whose sides are given by $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (1, 2, -3)$, $\mathbf{w} = (4, 0, 1)$