Lecture 13: Properties of Determinants

# 1 Cofactor Expansion Redux

Although we demonstrated cofactor expansions for calculating determinants of $4 \times 4$ matrices, they also work for smaller and larger matrices.

Example: $\begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 5 & 0 & 4 \end{vmatrix} = -11$ because

$$\text{Example: } \begin{vmatrix} 0 & 2 & 0 & 0 & 2 \\ 1 & 3 & 0 & -2 & 7 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 5 \\ 2 & 3 & 0 & 5 & 1 \end{vmatrix} = -36$$

## 2 Transposing Doesn't Change the Determinant

Example: $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

## 3 Row Operations and Determinants

Recall the three elementary row operations:

1. Multiply a row by a **nonzero** scalar $a$.
2. Exchange the order of two rows.
3. Add a scalar multiple of one row to another.

### 3.1 Scaling a SINGLE row by $a$ multiplies the determinant by $a$

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, and scale the second row by 7 to get $B = \begin{pmatrix} 1 & 0 & 2 \\ 7 \cdot 3 & 7 \cdot 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

**Observation:** Scaling a WHOLE $n \times n$ MATRIX by $a$ multiples the determinant by $a^n$ because we get one factor of $a$ for each of the $n$ rows.

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, and let

$B = 2A = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 10 & 0 \\ 2 & 2 & 0 \end{pmatrix}$.

### 3.2 Exchanging two rows negates the determinant

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, and switch the first and second rows to get $B = \begin{pmatrix} 3 & 5 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

**Observation:** If a matrix has two rows the same, then the determinant is zero

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$.

Let $B$ be the matrix you get when you switch the first two rows of $A$.

Then we know $\det(B) = -\det(A)$.

But $B = A$ since the first two rows of $A$ are the same

Hence $\det(A) = -\det(A)$.

## 3.3 If you add a scalar multiple of one row to another, the determinant doesn't change

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, and subtract the first row from the third to get $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 0 \\ 1-1 & 1-0 & 0-2 \end{pmatrix}$

## 4 Summary: Effect of Row Operations on Determinant

| Operation | Effect on Determinant |
|---|---|
| Scale a row by $a \neq 0$ | Scale by $a \neq 0$ |
| Switch two rows | Negate |
| Add a multiple of one row to another | Nothing |

Now we will look at sample elementary matrices for these three operations

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Now we have seen that

if $A$ is an $n \times n$ matrix

and $E$ is an $n \times n$ elementary matrix,

then $\det(EA) = \det(E)\det(A)$.

It turns out that this is part of a much bigger phenomenon.

Theorem: $\det(AB) = \det(A)\det(B)$ for any $n \times n$ matrices $A$ and $B$

A sketch of a proof of this fact is at the end of Section 4.2

Example: If $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ -2 & -4 \end{pmatrix}$,

$\det(AB) =$

## 5 Computing Determinants by Row Reduction

If you row reduce a matrix, keeping track of what each step does to the determinant, then you can compute the determinant.

For example, $\begin{vmatrix} 2 & -8 & 4 \\ -2 & 8 & -9 \\ -3 & 21 & 0 \end{vmatrix}$

## 6 Column Operations

| Operation | Effect on Determinant |
|---|---|
| Scale a column by $a \neq 0$ | Scale by $a \neq 0$ |
| Switch column rows | Negate |
| Add a multiple of one column to another | Nothing |

Example: $\begin{vmatrix} 2 & -8 & 4 \\ -2 & 8 & -9 \\ -3 & 21 & 0 \end{vmatrix}$

# 7 Determinant as a Criterion for Invertibility

Claim: The square matrix $A$ is invertible if and only if $\det(A) \neq 0$

(We already saw this for $2 \times 2$ matrices.)

Why is this true?

If $A$ is invertible, then

Example: $A = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix}$

If $A$ is not invertible, row reduce $A$ by $k$ elementary row operations (with elementary matrices $E_1, E_2, \ldots, E_k$) into its reduced row echelon form $R$.

$$E_k \cdots E_2 E_1 A = R$$

By the unifying theorem (Theorem 3.4.9 in the book):

Since $A$ is non-invertible
$R$ is not the identity matrix $I_n$

So $R$ has

Example: $A = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$