Lecture 12: Determinants and Cofactors

1 Determinants of 2×2 Matrices

Recall from Lecture 8, that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc = 0$, the matrix is not invertible

We called $ad - bc$ the **determinant** of A, written $\det(A)$ of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, and so we have the diagnostic:

- If $\det(A) \neq 0$, then A is invertible with inverse as given above
- If $\det(A) = 0$,then A is singular (non-invertible)

2 Determinants of 3×3 Matrices

It turns out that the notion of determinant can be defined for larger SQUARE matrices. So, for instance, the determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

3 Determinants of Larger Matrices

This pattern of using products of all (wrapping-around) diagonal stripes does NOT carry over to 4×4 and larger matrices.

For these we use a method called **cofactor expansion**, which is somewhat complicated. Our plan of attack.

1. Define the ijth minor of an $n \times n$ matrix
2. Define the ijth cofactor of an $n \times n$ matrix
3. Describe cofactor expansion along a row
4. Describe cofactor expansion along a column

4 Minors

If A is an $n \times n$ matrix, the ijth **minor** of A, often written M_{ij}, is the determinant of the $(n-1) \times (n-1)$ matrix that remains when we strike out the ith row and j column from A

Example: The $2,3$-minor M_{23} of

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \\ 6 & 1 & 0 & -2 \\ 0 & 0 & -1 & 3 \end{pmatrix} \text{ is}$$

5 Cofactors

If A is an $n \times n$ matrix, the ijth **cofactor** of A, often written C_{ij}, is $(-1)^{i+j}$ times the ijth minor of A, that is $C_{ij} = (-1)^{i+j} M_{ij}$

Example: The $2,3$-cofactor C_{23} of

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \\ 6 & 1 & 0 & -2 \\ 0 & 0 & -1 & 3 \end{pmatrix} \text{ is}$$

Now you can get the determinant of an $n \times n$ matrix A by picking any row of the matrix (say the ith row) and taking the following sum involving the entries of the ith row and the cofactors indexed by that row.

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

For example if $A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \\ 6 & 1 & 0 & -2 \\ 0 & 0 & -1 & 3 \end{pmatrix}$, we can use the cofactor expansion of the second row:

7 Cofactor Expansion along a Column

You can also get the determinant by picking any column (say the jth column) and and taking the analogous sum that involves entries of the j column and the corresponding cofactors

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

For example if $A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ -4 & 0 & 0 & 1 \\ 6 & 1 & 0 & -2 \\ 0 & 0 & -1 & 3 \end{pmatrix}$

8 Consequences for Diagonal and Triangular Matrices

Now we see that it is really easy to evaluate determinants of diagonal and triangular matrices.