Lecture 11: Geometry of Linear Systems

1 Span and Linear Systems

A question about span that becomes a question about consistency of a linear system:

Is $(3, 0, 1)$ in $\text{span}\{(1, 0, -2), (2, 1, -3), (1, 1, -1)\}$?

Is $(1, 2, 0)$ in $\operatorname{span}\{(1, 0, -2), (2, 1, -3), (1, 1, -1)\}$?

A question about linear dependence that becomes a question about NONTRIVIAL solutions of a homogeneous linear system:

Are $(1, 0, -2)$, $(2, 1, -3)$, and $(1, 1, -1)$ linearly dependent?

Are $(1, 0, -2)$, $(2, 1, -3)$, and $(1, 1, 0)$ linearly dependent?

Now recall from slide 2 that the general solution to

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is $(x, y, z) = (-3, 2, 0) + t(1, -1, 1)$.

and from slide 3 that the general solution to

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is $(x, y, z) = t(1, -1, 1)$.

3 Orthogonality Hyperplanes

If $\mathbf{v}$ is a vector in $\mathbb{R}^n$, we use $\mathbf{v}^\perp$ as a shorthand for the set of all vectors orthogonal to $\mathbf{v}$.

Examples: (i) $\mathbf{v} = (3, 4) \in \mathbb{R}^2$
(ii) $\mathbf{v} = (1, -1, 2) \in \mathbb{R}^3$.

A **hyperplane** in $\mathbb{R}^n$ is the set of solutions $(x_1, x_2, \ldots, x_n)$ to a single linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ are scalars, not all of them equal to zero.

The examples on the previous page have $b = 0$, so they include the point $(0, 0, \ldots, 0)$. Such hyperplanes are said to be **through the origin**.

Some hyperplanes not through the origin:

- Hyperplanes in $\mathbb{R}^2$ are
- Hyperplanes in $\mathbb{R}^3$ are
- Hyperplanes in $\mathbb{R}^4$ are

The solution to the system

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is just the intersection of three hyperplanes through the origin.

The solution to the system

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is just the intersection of three hyperplanes (two of them not through the origin)

4 Diagonal Matrices

A **diagonal matrix** is a square matrix where the entries not on the diagonal are all zero, for example

Sums and products of diagonal matrices are diagonal

A diagonal matrix with a zero on the diagonal is not invertible

Otherwise, you can easily find the inverse

So any power of a diagonal matrix is easy to calculate

5 Upper Triangular Matrices

An **upper triangular matrix** is a square matrix with all entries below the diagonal equal to zero, for example

Sums and products of upper triangular matrices are upper triangular

An upper triangular matrix with a zero on the diagonal is not invertible

If all diagonal entries of an upper triangular matrix are nonzero, then it is invertible with an upper triangular inverse

A **lower triangular matrix** is a square matrix with all entries above the diagonal equal to zero, for example

The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular

Lower triangular matrices have properties analogous to those of upper triangular matrices:

- Sums and products of lower triangular matrices are lower triangular
- A lower triangular matrix with a zero on the diagonal is not invertible
- If all diagonal entries of an lower triangular matrix are nonzero, then it is invertible with a lower triangular inverse

A **symmetric** matrix is a square matrix A with $A = A^T$, or equivalently $(A)_{ij} = A_{ji}$ for all i and j

Sum, differences, and scalar multiples of symmetric matrices are symmetric, but not always products

If a symmetric matrix A is invertible, then its inverse is symmetric, because we already have shown (Theorem 3.2.11) that if A is invertible, then A^T is also invertible with $(A^{-1})^T = (A^T)^{-1}$.