Lecture 10: Subspaces, Span, and Linear Independence

1 Subspaces

The set $\mathbb{R}^n$ of vectors with n entries is n-dimensional space.

A subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if

1. For any vectors $\mathbf{u}, \mathbf{v}$ in S, the sum $\mathbf{u} + \mathbf{v}$ is also in S (closure under addition)
2. For any vector $\mathbf{u}$ in S and any scalar $k \in \mathbb{R}$, the vector $k\mathbf{u}$ is also in S (closure under scalar multiplication)

Or we can equally well define a subspace of $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ following the single rule

- For any vectors $\mathbf{u}, \mathbf{v}$ in S and any scalars $k, \ell \in \mathbb{R}$, the vector $k\mathbf{u} + \ell\mathbf{v}$ is also in S (closure under linear combination)

Example Subspace 1: A line through the origin

A line passing through the origin in $\mathbb{R}^n$ has vector equation $\mathbf{x} = (x_1, \ldots, x_n) = t\mathbf{a}$ for $-\infty < t < \infty$ where $\mathbf{a}$ is a fixed vector giving the direction of the line

Then if we take two points on the line $\mathbf{x}_1 = t_1\mathbf{a}$ and $\mathbf{x}_2 = t_2\mathbf{a}$ and two scalars $k, \ell \in \mathbb{R}$, then

$$k\mathbf{x}_1 + \ell\mathbf{x}_2 = k(t_1\mathbf{a}) + \ell(t_2\mathbf{a}) =$$

which is also on the line.

So a line through the origin is a subspace of $\mathbb{R}^n$.

Example Subspace 2: The zero subspace of $\mathbb{R}^n$ is the set $\{\mathbf{0}\}$

Example Subspace 3: The full space $\mathbb{R}^n$ is, according to the definition, a subspace of $\mathbb{R}^n$

Examples 2 and 3 are called the **trivial subspaces** of $\mathbb{R}^n$

A Non-Subspace: A line not passing through the origin.

Consider the line $\mathbf{x} = t(1,2) + (1,0)$ in $\mathbb{R}^2$

Note that it does not pass through the origin.

This is not a subspace: take $\mathbf{u} = \mathbf{v} = (1,0)$

Then $\mathbf{u} + \mathbf{v} =$

But this is not on our line because

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is a collection of vectors in $\mathbb{R}^n$, we define the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ to be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

That $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is the set of all vectors of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

where $a_1, a_2, \ldots, a_n$ are scalars.

Example: A line through the origin is given by $\mathbf{x} = t\mathbf{v}$ for all scalars t: it is the span of a single nonzero vector $\mathbf{v}$.

Example: A plane through the origin is given by $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for all scalars s, t: it is the span of two non-parallel vectors $\mathbf{u}$, $\mathbf{v}$

The span of any collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of vectors is automatically a subspace:

And if any subspace contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, it must contain the entire span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$

Any subspace is a span of a set of vectors

The only subspaces of $\mathbb{R}^2$ are

- the zero subspace
- the lines through the origin
- the full space $\mathbb{R}^2$

The only subspace of $\mathbb{R}^3$ are

-
-
-
-

General points, lines, and planes (not necessarily containing the origin) have vector equations of the form $\mathbf{x} = \mathbf{p}$, $\mathbf{x} = \mathbf{p} + t\mathbf{v}_1$ and $\mathbf{x} = \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2$, respectively (with $-\infty < s, t < \infty$)

These are obtained by translating by $\mathbf{p}$ the subspaces described by $\mathbf{x} = \mathbf{0}$, $\mathbf{x} = t\mathbf{v}_1$, and $\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$.

These can be thought of as

$$\mathbf{x} = \mathbf{p} + \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\},$$

and if we let $k > 2$, we can get higher-dimensional objects than planes.

These are known as **affine spaces**.

They are only subspaces if they pass through the origin.

Suppose A is an $m \times n$ matrix and we have a homogeneous linear system

$$A\mathbf{x} = \mathbf{0}$$

(so $\mathbf{x}$ is $n \times 1$ and $\mathbf{0}$ is $m \times 1$)

The solution set S for our system is a subspace of $\mathbb{R}^n$ because if $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$, then

We call this subspace S containing all solutions to $A\mathbf{x} = \mathbf{0}$ the **solution space** for the system

For A an $m \times n$ matrix, the solution space of $A\mathbf{x} = \mathbf{0}$ is $\mathbb{R}^n$ if and only if $A = 0_{m \times n}$.

Clearly if $A = 0$, then any $\mathbf{x}$ solves $0\mathbf{x} = \mathbf{0}$, so the solution space is all of $\mathbb{R}^n$

Conversely, if $A \neq 0$, then it has a column that is not all zeroes, say the jth column $\mathbf{c}_j(A) \neq \mathbf{0}$. Let $\mathbf{e}_j$ be our usual standard vector with 1 in the jth position and zeroes everywhere else. Then

$A\mathbf{e}_j =$

which is nonzero, so $\mathbf{e}_j$ doesn't solve our system, and so the solution space is not all of $\mathbb{R}^n$.

We can now make a criterion for equality of matrices based on how the multiply with vectors

If A and B are $m \times n$ matrices, then $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $A = B$

If $A = B$, then clearly $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$

Conversely, if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, then

A collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of vectors is said to be **linearly dependent** if there are some scalars $a_1, a_2, \ldots, a_m$, at least one of which is nonzero, such that

(1)

An equation of this form where at least one scalar a_i is nonzero is called a **dependence relation** for $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

The collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is **linearly independent** if it is not linearly dependent, i.e., if the only choice of scalars $a_1, a_2, \ldots, a_m$ that satisfies (1) is the trivial one $a_1 = a_2 = \cdots = a_m = 0$.

Observation 1: Note that any collection that includes the zero vector, say $\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m$, must be be linearly dependent because

$$1\mathbf{0} + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_m = \mathbf{0}$$

is a dependence relation. (The scalars $1, 0, 0, \ldots, 0$ are not all zero.)

Observation 2: A single nonzero vector $\mathbf{v}$ is linearly independent since $a\mathbf{v} = \mathbf{0}$ only if $a = 0$

Observation 3: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if they are parallel:

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then either

- $\mathbf{u} = a\mathbf{v}$ for some scalar a, or
- $\mathbf{v} = b\mathbf{u}$ for some scalar b.

These can be rearranged into dependence relations

So they are dependent.

Conversely, if $\mathbf{u}$ and $\mathbf{v}$ are dependent, then there is some dependence relation

$$c\mathbf{u} + d\mathbf{v} = \mathbf{0}$$

with at least one of c or d nonzero.

A collection $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of vector is linearly dependent if and only if one of these vectors can be expressed as a linear combination of the rest.

If one vector, say $\mathbf{v}_j$, is a linear combination of the rest

$$\mathbf{v}_j = a_1\mathbf{v}_1 + \ldots + a_{j-1}\mathbf{v}_{j-1} + a_{j+1}\mathbf{v}_{j+1} + \ldots + a_m\mathbf{v}_m$$

So $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

Conversely, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, write a dependence relation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_m\mathbf{v}_m = \mathbf{0}$$

At least one scalar must be nonzero: let j be its index.

For an $m \times n$ matrix A, the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if the columns $\mathbf{c}_1(A), \mathbf{c}_2(A), \ldots, \mathbf{c}_n(A)$ are linearly dependent.

This is because if we write $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then

So nontrivial solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to $A\mathbf{x} = \mathbf{0}$ is a choice of $x_1, x_2, \ldots, x_n$, at least one of which is nonzero, such that

$$x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + \cdots + x_n\mathbf{c}_n(A) = \mathbf{0}$$

which is

Claim: Any collection of more than n vectors in $\mathbb{R}^n$ must be linearly dependent.

Proof: Call the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ with $k > n$, and make a matrix A whose columns are the vectors

Then A is $n \times k$, and so has fewer rows than columns

Thus the homogeneous system $A\mathbf{x} = \mathbf{0}$ has more

So by a principle discussed in Lecture 5, it has

So, by what we just discussed on the previous slide, the columns $\mathbf{a}_1, \ldots, \mathbf{a}_k$ of A are

6 An Even Bigger Summary

If A is an $n \times n$ square matrix, the following are equivalent:

(i). The reduced row echelon form of A is I_n

(ii). A is a product of elementary matrices

(iii). A is invertible

(iv). $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(v). $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$

(vi). $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b} \in \mathbb{R}^n$

(vii). The columns of A are linearly independent

(viii). The rows of A are linearly independent

The equivalence of Items (i)–(vi) was shown in Lecture 9

The equivalence of Items (iv) and (vii) was shown on slide 15 of this lecture.

To finish, we show that Item (viii) is equivalent to Item (iv)

For Item (viii) since the rows of A are the columns of

Now A^T is invertible if and only if A is invertible by slide 18 of Lecture 8