Lecture 9: Elementary Matrices, Finding Inverses

1 Row Operations and Elementary Matrices

Recall the three row operations:

1. Multiply a row by a **nonzero** scalar.
2. Exchange the order of two rows.
3. Add a multiple of one row to another.

To each row operation there corresponds a square matrix called an **elementary matrix**.

Examples: Suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To multiply the second row by -3, let

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ so that

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To exchange the first row with the third row,

set $F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ so that

$$FA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To add -2 times the first row to the second row,

set $G = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so that

$$GA = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

To do these three steps in order (first scale the second row by -3, then swap the first and third rows, and finally add -2 times the first row to the second) use

Note that the first operation is the one whose matrix first gets multiplied with A, so it is the **rightmost** matrix.

Also note that since every row operation can be undone with another row operation, any elementary matrix is invertible and its inverse is another elementary matrix

Examples: scaling the second row by -3 is opposite to scaling the second row by $-1/3$, and you can check that

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exchanging the first and third rows is undone by just exchanging them back again, so F is its own inverse! You can check that $FF = I_3$.

Adding -2 times the first row to the second row is opposite to adding 2 times the first row o the second row, and you can check that

$$G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition: two matrices are **row equivalent** if one can be obtained from the other by performing a sequence of row operations

We can equally well say that A is row equivalent to B if there are some elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = B$.

Now suppose that A is a square $n \times n$ matrix and we row-reduce it to its reduced row echelon form R by means of k row operations corresponding to elementary matrices $E_1, E_2, \ldots, E_k$.

$$E_k \cdots E_2 E_1 A = R$$

So R is row equivalent to A, so of course R is the same size as A, that is, $n \times n$, since row reduction preserves size.

If R has n leading ones, then since it only has n rows and columns, the only way to arrange the leadings ones is like

$$R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

So then

Since A is a product of elementary matrices, each invertible, A is invertible with inverse

OK. So if the reduced row echelon form R of A has n leading ones, then $R = I_n$ and A is invertible.

What's more, we can figure out the inverse of A from the row operations we performed.

A neat way to do this is to first build the $n \times (2n)$ matrix obtained by writing A and I_n side-by-side $\left(A \middle| I_n\right) =$

$$\left(\begin{array}{cccc|cccc} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & 0 & \cdots & 1 \end{array}\right)$$

Then apply the elementary row operations that take A to I_n, that is

$$E_k \cdots E_2 E_1 \left(A \middle| I_n\right)$$

which is just

$$\left(E_k \cdots E_2 E_1 A \middle| E_k \cdots E_2 E_1 I_n\right)$$

and these row operations take A to I_n, so we have

$$\left(I_n \middle| E_k \cdots E_2 E_1\right)$$

and now notice that the right hand matrix has become $E_k E_2 \cdots E_1 =$

Example: to compute the inverse of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, write

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{array}\right)$$

and row reduce

$$\left(\begin{array}{cc|cc} & & & \\ & & & \end{array}\right)$$

$$\left(\begin{array}{cc|cc} & & & \\ & & & \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right)$$

So $A^{-1} =$

In summary: if we row reduce an $n \times n$ matrix A to get its reduced row echelon form R and R has n leading ones, then R must be I_n, and A is invertible, and we write both A and A^{-1} as products of elementary matrices.

Now let's see what happens if R has less than n leading ones.

Then not every row of R can have a leading one.

So R has at least one zero row

$$R = \begin{pmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We claim that R has no inverse

To show this, let us see what absurd things will happen if we assume R **does** have an inverse S

$$\begin{aligned}\begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix} &= \begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix} I \\ &= \begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix} RS\end{aligned}$$

which is absurd indeed! So our assumption that R is invertible must be false, and so R must **not** be invertible.

Recall that we got R from A by row operations

$$E_k \cdots E_2 E_1 A = R$$

and elementary matrices $E_1, \ldots, E_k$ for row operations are invertible.

If A were invertible, then R would be a product of invertible matrices, hence invertible. But we know that R is not invertible!

So A must not be invertible!

Summary for both cases: if we row reduce an $n \times n$ matrix A to get its reduced row echelon form R and R has n leading ones, then R must be I_n, and A is invertible, and we can write both A and A^{-1} as products of elementary matrices. If R has less than n leading ones, then A is not invertible.

3 The Big Summary

If A is an $n \times n$ square matrix, the following are equivalent:

(i). The reduced row echelon form of A is I_n

(ii). A is a product of elementary matrices

(iii). A is invertible

(iv). $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(v). $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$

(vi). $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b} \in \mathbb{R}^n$

The equivalence of items (i)–(iii) are from the "Summary of both cases" above.

The equivalence of these with items (iv) and (v) came from our observations in Lecture 8 that if A is invertible then $A\mathbf{x} = \mathbf{b}$ has a unique solution; this unique solution must be the trivial solution if $\mathbf{b} = \mathbf{0}$.

Item (v) certainly implies Item (vi), since having one solution makes the system consistent.

We now show that Item (vi) implies Item (i)

Consider the standard unit vector

$$\mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and let $E_k, \cdots, E_2, E_1$ be elementary matrices such that

$$E_k \cdots E_2 E_1 A = R$$

where R is the reduced row echelon form of A.

Then $A\mathbf{x} = (E_k \cdots E_2 E_1)^{-1}\mathbf{e}_n$, represented by

$$\left(A \middle| (E_k \cdots E_2 E_1)^{-1}\mathbf{e}_n\right)$$

has a solution.

We can row reduce it by applying our elementary matrices:

This system has a solution, since we are assuming Item (vi)

That would be impossible if the last row of R were zero

So R must have n leading ones, so $R = I_n$: we have shown that Item (vi) implies Item (i)

4 Checking Invertibility is Easier Now

If A and B are square $n \times n$ matrices with $AB = I_n$, then A and B are inverses of each other.

(That is, we no longer need to check $BA = I_n$ also!)

Proof: Suppose $AB = I_n$

Then if $\mathbf{b}$ is any vector in $\mathbb{R}^n$ and we are asked to solve the system

$$A\mathbf{x} = \mathbf{b}$$

So $A\mathbf{x} = \mathbf{b}$ is

Consulting the Big Summary, we see that

5 Solving Multiple Linear Systems at the Same Time

If we are asked to solve two linear systems involving the same matrix, say

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

then just row-reduce the augmented matrix

$$\left(\begin{array}{cc|c|c} 1 & 1 & -2 & 5 \\ -1 & 1 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c|c} & & & \\ & & & \end{array}\right)$$

$$\left(\begin{array}{cc|c|c} & & & \\ & & & \end{array}\right)$$

$$\left(\begin{array}{cc|c|c} 1 & 0 & -5/2 & 2 \\ 0 & 1 & 1/2 & 3 \end{array}\right)$$

6 The Consistency Problem

For a given $m \times n$ matrix A, this is the problem of finding which vectors $\mathbf{b} \in \mathbb{R}^m$ make the linear system

$$A\mathbf{x} = \mathbf{b}$$

consistent.

We now know that if A is an invertible matrix, any $\mathbf{b}$ makes the system consistent.

If A is non-invertible, we can use our row-reduction techniques

Example: If $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$, find which $\mathbf{b} \in \mathbb{R}^3$ make $A\mathbf{x} = \mathbf{b}$ consistent.

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 \\ 1 & 1 & 4 & b_3 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 2 & 4 & b_3 - b_1 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - 2b_2 \end{array}\right)$$

$$\left(\begin{array}{rrr|r} 1 & 0 & 2 & b_1 + b_2 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & b_3 - b_1 - 2b_2 \end{array}\right)$$

Solution

$$\begin{aligned} x_1 &= -2x_3 + b_1 + b_2 \\ x_2 &= -2x_3 + b_2 \\ 0 &= b_3 - b_1 - 2b_2 \end{aligned}$$

We must have $b_1 + 2b_2 - b_3 = 0$ for consistency, and that is sufficient.

(Because then we can set the free variable x_3 to whatever we want, and then the pivot variables x_1 and x_2 are determined.)