Lecture 5: Row Reduction

1 Gauss Elimination

A scheme for applying row operations

(pp. 52–53 of the text describes the algorithm)

We are going to illustrate it by actually doing it

$$\left(\begin{array}{cccccc|c} 0 & 0 & 0 & -1 & -3 & 0 & -7 \\ 0 & -2 & -10 & 4 & -8 & -14 & -20 \\ 0 & 3 & 15 & -8 & 9 & 30 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccccc|c} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array}\right)$$

$$\left(\begin{array}{c|c} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right)$$

$$\left(\begin{array}{c|c} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right)$$

$$\left(\begin{array}{c|c} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right)$$

$$\left(\begin{array}{c|c} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right)$$

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & -2 & 4 & 7 & 10 \\ 0 & 0 & 0 & 1 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The last matrix is said to be in **row echelon form**.

- all-zero rows at the bottom
- any nonzero row starts with a **leading one**
- leading ones occur further to the right as we move downward

The leadings ones are in **pivot positions**

2 Gauss-Jordan Elimination

First apply Gauss Elimination (forward phase)

Then clear the nonzero values above the pivot positions (backward phase)

$$\left(\begin{array}{c|c} \qquad\qquad\qquad\qquad\qquad\qquad & \qquad \\ \\ \\ \\ \end{array}\right)$$

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & -6 \\ 0 & 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This last matrix is in **reduced row echelon form**: it must satisfy **all the rules** for row-echelon form, plus the **additional rule**

- a column with a leading one (a **pivot column**) must have all entries equal to zero except the leading one.

Depending on choices you make during Gauss elimination, you might get **different row echelon** forms from the same starting matrix, but

There is **only one reduced row echelon** form you can get starting from a particular matrix, no matter what choices you make during the Gauss-Jordan elimination

3 Checking Consistency

If we have a slightly different system

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & -6 \\ 0 & 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

the last line represents the equation

which means the system is inconsistent.
In our system

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & -6 \\ 0 & 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

the last line represents the equation

so we may ignore it for the purposes of finding solutions. But if someone asks you for the reduced row echelon form of the original matrix $\left(\begin{array}{cccccc|c} 0 & 0 & 0 & -1 & -3 & 0 & -7 \\ 0 & -2 & -10 & 4 & -8 & -14 & -20 \\ 0 & 3 & 15 & -8 & 9 & 30 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ your answer **must** include the all-zero row because "delete the all zero row" is not a valid row operation.

4 Leading (Pivot) and Free Variables

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & -6 \\ 0 & 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \end{array}\right)$$

represents the system

solve for the **leading (pivot) variables**

The other variables are **free variables**.

They can be set in any way, and once they are chosen, the pivot variables are determined by them.

The textbook likes to give free variables new names to emphasize the fact that they are free parameters.

Number of Solutions

	Consistent	Inconsistent
0 Free Vars.	1	0
≥ 1 Free Vars.	∞	0

Eg., a consistent system without free variables

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right)$$

5 When You are Guaranteed to Have Free Variables

We get one pivot variable for each nonzero row of our reduced row echelon form

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & -6 \\ 0 & 0 & 0 & 1 & 0 & -9 & -2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\#$ Pivot Vars. $\leq$ $\#$ Equations

$\#$ Total Vars. $=$ $\#$ Free Vars. $+$ $\#$ Pivot Vars.

If there are more total variables than equations, then we are **guaranteed** to have free variables

$$\left(\begin{array}{cccccc|c} 0 & 1 & 5 & 0 & 0 & -23 & 0 \\ 0 & 0 & 0 & 1 & 0 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

All homogeneous systems are consistent, because they have the **trivial solution** where all the variables are zero.

For example, in this case

So a homogeneous system with more unknowns that equations is guaranteed to have infinitely many solutions (including the trivial solution, and infinitely many non-trivial solutions), because...

Number of Solutions

	Consistent	Inconsistent
0 Free Vars.	1	0
≥ 1 Free Vars.	∞	0

- Homogeneity guarantees that the system is consistent
- Having more unknowns than equations guarantees that we have at least one free variable.

For inhomogeneous systems, we can be in any of the four cells of the chart.

WARNING: If you have more equations than variables, you can not jump to the conclusion that you have no free variables: this system with 3 equations and 2 variables

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -3 \\ 0 & 5 & -5 \end{pmatrix}$$

has reduced row echelon form

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so has one free variable.