Lecture 4: Linear Systems

1 Linear Equations

A **linear equation** in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the form

where $a_1, a_2, \ldots, a_n$ and b are constants.

The unknowns can not appear with any powers (other than the first power), nor any functions like square root or sine or cosine applied to them

For example if x, y, z are the unknowns and a, b, c, d are constants

$$2x - 3y + 7z = 4$$

$$2x + 7z = 4$$

$$x + y + z - 2 = 0$$

$$ax + by + cz = d$$

$$ax + by^2 + cz = d$$

$$ax + b^2y + cz = d$$

$$ae^x + by + cz = d$$

$$x\sin(a) + by + cz = d$$

$$xy + by + cz = d$$

2 Homogeneous Linear Equations

A linear equation in $x_1, x_2, \ldots, x_n$ is said to be **homogeneous** if it only has terms of the form $a_j x_j$ and no purely constant term.

For example, if x, y, z are unknowns

$$2x - 3y + 7z = 4$$

$$2x - 3y + 7z = 0$$

$$x + z - 1 = 0$$

$$x + y = 0$$

3 Linear Systems

A collection of linear equations is called a **linear system**

$$\begin{aligned} y + z &= 5 \\ 2x + 3y + z &= 0 \end{aligned}$$

And a collection of linear equations, all of which are homogeneous, is called a **homogeneous linear system**

$$\begin{aligned} y + z &= 0 \\ 2x + 3y + z &= 0 \end{aligned}$$

A **solution** to a linear system is an assignment of values to the unknowns that satisfies all the equations.

So $(x, y, z) = (15, 5, 0)$ and $(x, y, z) = (7, 1, 4)$ are both solutions of

$$\begin{aligned} y + z &= 5 \\ -x + 3y + z &= 0 \end{aligned}$$

Note that a homogeneous linear system like

$$\begin{aligned} y + z &= 0 \\ 2x + 3y + z &= 0 \end{aligned}$$

always has at least one obvious solution, namely $(x, y, z) = (0, 0, 0)$.

This system also has other solutions like

5 Consistency

A system with at least one solution is called **consistent**

A system with no solutions at all is called **inconsistent**, for example

$$\begin{aligned} x + y + z &= 0 \\ 2x + 2y + 2z &= 1 \end{aligned}$$

We use matrices to represent linear systems. If the unknowns are $x_1, x_2, \ldots, x_n$ write the linear system in the form

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m,
\end{aligned}$$

where the $a_{i,j}$'s and b_i's are constants.

Then the augmented matrix is

$$\left(\begin{array}{cccc|c}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m
\end{array}\right)$$

For example

$$\begin{aligned} y + z &= 5 \\ -x + 3y + z &= 0 \end{aligned}$$

in matrix form is

$$\left(\begin{array}{ccc|c} & & & \\ & & & \end{array}\right)$$

Solving a system At each step, modify one equation and **keep all the others**

The **wrong way**:

$$
\begin{aligned}
7x - 5y &= 2 \\
2x + 5y &= 5 \\
x + y &= 1
\end{aligned}
$$

The **right way** and the **matrix way**:

$$\begin{aligned} 7x - 5y &= 2 \\ 2x + 5y &= 5 \\ x + y &= 1 \end{aligned}$$

7 Elementary row operations:

1. Multiply an equation by a **nonzero** scalar.
2. Exchange the order of two equations.
3. Add a multiple of one equation to another. (Or, equivalently, subtract a multiple of one equation from another.)

These are used because they **don't change** the solutions.

8 Solution Sets

The **solution set** of a linear system is the set of all solutions.

$$\begin{aligned} 2x + 5y &= 5 \\ x + y &= 1 \end{aligned}$$

$$
\begin{aligned}
6y - 3z &= 3 \\
-2y + z &= -1
\end{aligned}
$$

$$
\begin{aligned}
6x - 3y &= 4 \\
-2x + y &= -1
\end{aligned}
$$