Lecture 3: Equations for Lines and Planes

1 Slope-Intercept Equation for a Line

The most familiar way to describe a line in $\mathbb{R}^2$ is the **slope-intercept form** $y = mx + b$, where m is the slope and b is the y-intercept, for example: $y = \frac{1}{2}x - 1$.

We can describe all lines in the plane this way, except vertical ones, which have the form $x = c$, with c a constant. For example: $x = 2$.

To avoid having two different forms, we use **general equations** of lines, which have the form $ax + by = c$.

For example, we can rearrange the equation for our first line $y = \frac{1}{2}x - 1$ to be

So more than one general equation can describe the same line.

Our vertical line $x = 2$ is already in general form, since it is just $1x + 0y = 2$.

3 Describing Lines in $\mathbb{R}^2$ Using Inner Product

A line in $\mathbb{R}^2$ that passes through the origin can be described using vectors. If $\mathbf{n} = (a, b)$ is a vector, consider all the vectors $\mathbf{r} = (x, y)$ orthogonal to $\mathbf{n}$. Take $\mathbf{n} = (-1, 2)$ as an

The set of $\mathbf{r} = (x, y)$ with $\mathbf{n} \cdot \mathbf{r} = 0$ is

We say that $\mathbf{n}$ is a **normal vector** for that line.

We can also have a line in the same direction (so it has the same normal $\mathbf{n} = (-1, 2)$), but not through the origin.

Example, the line that passes through $\mathbf{p} = (2, 0)$ with normal $\mathbf{n} = (-1, 2)$ is given by **point-normal form**

Point-normal forms are not unique (because the point $\mathbf{p}$ can be any point on the line, and the normal $\mathbf{n}$ can be replaced with a nonzero scalar multiple of itself), so if you try to make our general form back into a point-normal form, you may get a different point-normal form than the one you started with.

For instance $-x+2y = -2$ can be rearranged to $0 = -(x-2)+2(y-0)$ or $(x-0)-2(y+1) = 0$

If we can describe a line by a normal (perpendicular) vector, we also can describe it by a parallel vector.

For instance, consider all the points of the form $(2t, t)$ (with $-\infty < t < \infty$): they form a line through the origin:

We can consider this line the set of vectors $\mathbf{r} = (x, y) = t\mathbf{v}$ with $\mathbf{v} = (2, 1)$: the equation $\mathbf{r} = (x, y) = t(2, 1)$ is the **vector equation** of our line through the origin

If we take a vector equation $\mathbf{r} = (x, y) = (2t, t)$ and write a separate scalar equation for each component

then the system of scalar equations are known as a set of **parametric equations** of the line.

Vector and parametric equations are not unique, since using a nonzero scalar multiple of $\mathbf{v}$ will produce the same line: $t(-4, -2)$ sweeps through the same points as t runs through $\mathbb{R}$.

If we want a line that's not through the origin, but is through a point $\mathbf{p}$, we just translate these vectors $t\mathbf{v}$ by the vector $\mathbf{p}$. So if $\mathbf{p} = (2, 0)$, we get

We can recover a general equation from a set of parametric equations $x = 2 + 2t$, $y = t$ by eliminating the variable t.

And we can go from a general equation $ax + by = c$ to parametric equations: solve for one of the variables (x or y) in terms of the other variable (y or x) and then set t equal to the other variable.

For example: $x - 2y = 2$

Note that the vector $\mathbf{v} = (2, 1)$ from a vector equation $\mathbf{x} = t\mathbf{v} + \mathbf{p}$ will be orthogonal to a normal vector $\mathbf{n}$ from a point-normal equation $\mathbf{n} = (-1, 2)$.

5 Application: A Line Through Two Points

The line passing through $(1, 4)$ and $(-3, 2)$ has direction parallel to

And, as we just said, it passes through $(1, 4)$, so it has vector equation

So it has general equation

6 General Equation of a Plane in $\mathbb{R}^3$

A **general equation** for a **plane** in $\mathbb{R}^3$ looks like a general equation for a line in $\mathbb{R}^2$, except it has one more variable:

$$ax + by + cz = d$$

For example, the plane $z = 3$

Or the plane $x + y + z = 1$

7 Point-Normal Equation

Again, this is analogous to that for a line in $\mathbb{R}^2$: we will have a point $\mathbf{p}$ on the plane and a vector $\mathbf{n}$ that is perpendicular to all the directions of lines in the planes.

Then the point-normal form is

Example: $\mathbf{p} = (0, 0, 3)$, $\mathbf{n} = (0, 0, 1)$

Example: $\mathbf{p} = (1, 0, 0)$, $\mathbf{n} = (1, 1, 1)$

Since planes are two-dimensional, you can walk in lots of different directions.

You need two vectors to describe all these.

This is where the concept of **linear combinations** comes up again.

You need to find two vectors $\mathbf{v}$ and $\mathbf{w}$ that are not parallel to each other, each describing a direction in the plane.

The plane through the origin with directions $\mathbf{v}$ and $\mathbf{w}$ is given by all the linear combinations of $\mathbf{v}$ and $\mathbf{w}$:

where $-\infty < s, t < \infty$.

Example: If $\mathbf{v} = (1, 0, 0)$ and $\mathbf{w} = (0, 1, 0)$, then the plane through the origin given by all linear combinations $\mathbf{r} = (x, y, z) = s\mathbf{v} + t\mathbf{w}$ is

This is a **vector equation** of the plane. If you separate it into components you get a set of **parametric equations**.

If we want a vector equation of a plane that does not pass through the origin, we can translate these points by a vector $\mathbf{p}$ to get

Example: $\mathbf{v} = (1, 0, 0)$, $\mathbf{w} = (0, 1, 0)$, and $\mathbf{p} = (0, 0, 3)$

You can recover a general equation from a vector equation by eliminating the variables s and t. For instance, if we have $(x, y, z) = s\mathbf{v} + t\mathbf{w} + \mathbf{p}$ with $\mathbf{v} = (1, -1, 0)$, $\mathbf{w} = (0, 1, -1)$ and $\mathbf{p} = (0, 0, 1)$, then

$$(x, y, z) = s(1, -1, 0) + t(0, 1, -1) + (0, 0, 1)$$

We can get a vector equation from a general equation $ax+by+cz = d$ by the reverse process of introducing variables s and t.

Solve for one of the three variables in terms of the other two. For example, if a general equation is $x+y+z = 1$, we can solve for x in terms of y and z.

Then define $s = y$ and $t = z$, to get

Or, suppose we have the plane $z = 3$; then we still can write $z = 0x + 0y + 3$.

And set $s = x$ and $t = y$ to get

Geometric Application: Plane Through Three Points Three points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$ define a plane if the points are **non-collinear**.

Example: $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (0, 0, 1)$

Check that $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ are not parallel vectors.

So our plane is the plane through $\mathbf{a}$ with directions $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$.