Lecture 2: Inner Product, Angles, and Orthogonality

Math 232: Applied Linear Algebra
Simon Fraser University

1 Review of Geometry: Distance Between Two Points

If $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ are points in $\mathbb{R}^n$, then the vector $\mathbf{u}$ with initial point A and terminal point B is just

And the distance $d(A, B)$ between A and B is $||\mathbf{u}|| =$

Example: the distance between $(1, 1, 1)$ and $(-1, -2, -3)$

Of course we have for any A and B

- $d(A, B) = d(B, A)$
- $d(A, B) \geq 0$
- $d(A, B) = 0$ if and only if $A = B$

2 Unit Vectors, Normalization

A vector of length 1 is called a **unit vector**.

If $\mathbf{u}$ is a nonzero vector, then the unit vector with the same direction as $\mathbf{u}$ is

We call this procedure **normalizing u**.

When working in two dimensions, we have the **standard unit vectors**

Now our notion of linear combinations, mentioned last time, comes in handy: every vector in $\mathbb{R}^2$ is a linear combination of the standard unit vectors.

Example: $(-4, 3) =$

and in three dimensions, the standard unit vectors are

and so $(-6, 0, 7) =$

In 672 dimensions, we can no longer afford to waste one letter per vector. So in a generic setting of n dimensions, we instead write

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \ldots, 0, 0) \\
\mathbf{e}_2 &= (0, 1, 0, \ldots, 0, 0) \\
\cdots \\
\mathbf{e}_n &= (0, 0, 0, \ldots, 0, 1)
\end{aligned}$$

So, if $n = 5$, then $(-3, 1, 0, 8, 2) =$

3 Inner Product and Length

The **inner product** or **dot product** of two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is

Notice that the dot product takes a **pair of vectors** as inputs, and the output is a **scalar**.

$(-2, 1, 3) \cdot (4, 0, 7) =$

Notice that $\mathbf{u} \cdot \mathbf{u}$ is just the square of the length of $\mathbf{u}$:

So $\mathbf{u} \cdot \mathbf{u} \geq 0$ **always**, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

4 Algebraic Rules for Inner Products

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and a is a scalar,then

1). $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

2). $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

3). $(a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v})$

From these we can get lots of other rules, for example,

4). $\mathbf{0} \cdot \mathbf{u} = 0$

5). $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

6). $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

7). $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

8). $\mathbf{u} \cdot (a\mathbf{v}) = a(\mathbf{u} \cdot \mathbf{v})$

5 Inner Product and Angle

The angle between two nonzero vectors:

- put both vectors at the same initial point
- the angle between them, θ, is just the angle of a rotation that would place the one on top of the other (go the shortest way: $0 \leq \theta \leq \pi$)

The relation between angle and inner product is

Example: the angle between $\mathbf{u} = (-1, 4, -1)$ and $\mathbf{v} = (-3, -3, 0)$ is

6 Orthogonality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal**, and we write $\mathbf{u} \perp \mathbf{v}$, if

Note that $\mathbf{0}$ is orthogonal to any other vector

If both vectors are nonzero, $\mathbf{u} \cdot \mathbf{v} = 0$ means that the angle between $\mathbf{u}$ and $\mathbf{v}$ is

Note that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $||\mathbf{u} + \mathbf{v}||^2 =$

More generally, a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is said to be orthogonal if **every pair** of them is orthogonal.

Example: $\mathbf{u} = (-1, 2, 1), \mathbf{v} = (1, 0, 1), \mathbf{w} = (-1, -1, 1)$

A set of vectors is **orthonormal** if it is an orthogonal collection and if, in addition, every vector is a unit vector.

Example: our orthogonal set above is **not** orthonormal

7 The Cauchy-Schwarz Inequality

Recall that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta,$$

where θ is the angle between the two vectors.

Now $||\mathbf{u}||, ||\mathbf{v}|| \geq 0$ and $-1 \leq \cos\theta \leq 1$, so

8 The Triangle Inequality

Recall that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

For general $\mathbf{u}$ and $\mathbf{v}$ (not necessarily orthogonal), we have
$||\mathbf{u}+\mathbf{v}||^2 =$

This implies that for any points A, B, and C, we have $d(A,C) \leq d(A,B) + d(B,C)$

9 Geometric Application: The Parallelogram Law

Suppose we have a parallelogram with two sides given by vectors $\mathbf{u}$ and $\mathbf{v}$

Then $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 =$