Lecture 1: Space and Vectors

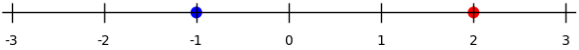
Math 232: Applied Linear Algebra
Simon Fraser University

05 September 2012

1-Dimensional Space

Algebraically: $\mathbb{R}$, the set of all real numbers (finite and infinite decimal expansions)

Geometrically: the number line



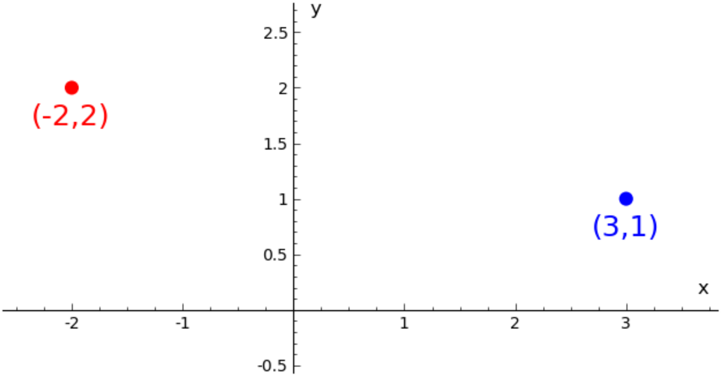
The red point represents 2 and the blue point represents -1.

To assign coordinates to points on a line, we need to fix a zero point, decide which way is positive, and set a unit of measure.

2-Dimensional Space

Algebraically: $\mathbb{R}^2$, the set of all ordered pairs (x, y) of real numbers

Geometrically: the plane

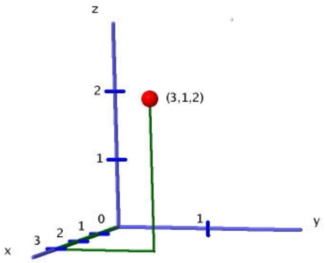


To assign coordinates to points in the plane, we have set a zero point, assigned compass directions, and set units of measure along each direction.

3-Dimensional Space

Algebraically: $\mathbb{R}^3$, the set of all ordered triples (x, y, z) of real numbers

Geometrically: "solid" space



(Red point is at $(3, 1, 2)$. Green lines show that we can get there by going 3 units in the positive x-direction, then 1 unit in the positive y-direction, and 2 units in the positive z-direction.)

This is a right-handed coordinate system.

n-Dimensional Space ($n \geq 1$)

Algebraically: $\mathbb{R}^n$, the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers

Geometrically: not visualizable for large n.

Scalars

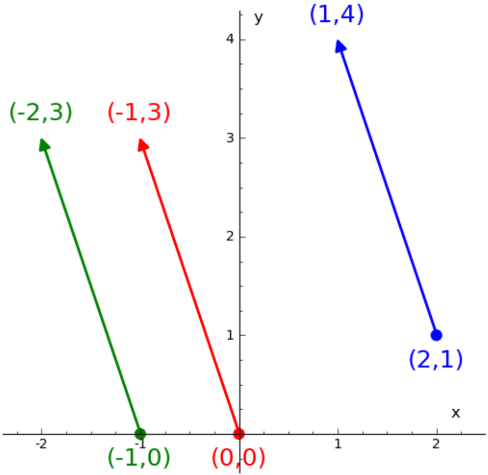
A *scalar* is just a number

For now, we work with real scalars, that is, elements of $\mathbb{R}$

Later, we will consider imaginary and complex numbers

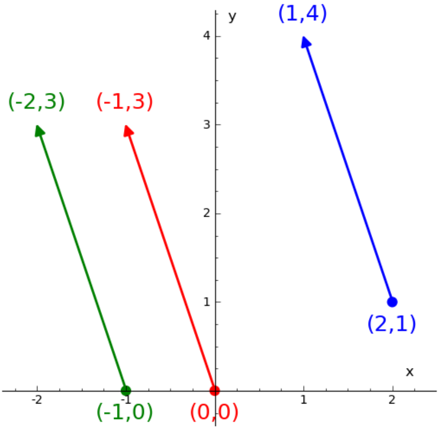
Vectors in 2 Dimensions, Geometrically

Geometrically, can be thought of as a *displacement* in the plane: a difference in position, having both *distance* and *direction*.



These three vectors all correspond to a displacement of one unit to the left and three units upward, so they are all the *same* (or *equal*, or *equivalent*) vector.

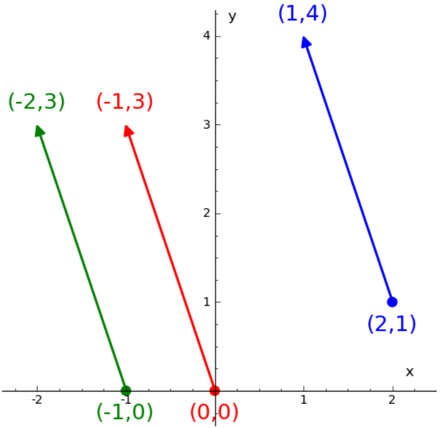
Initial and Terminal Points



We call the vector $(-1, 3)$ since the displacement is one unit to the left and three units upward.

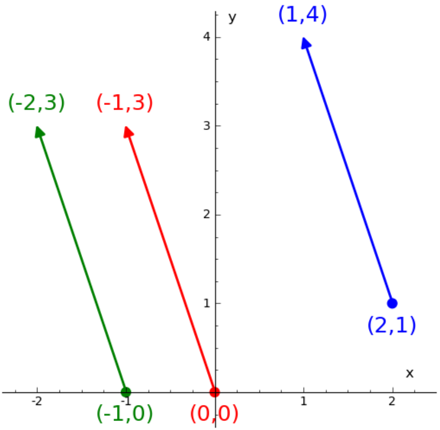
In general, if (u, v) is the vector with initial point (x_i, y_i) and terminal point (x_t, y_t), then $(u, v) = (x_t - x_i, y_t - y_i)$.

Calculating a Vector Initial and Terminal Points



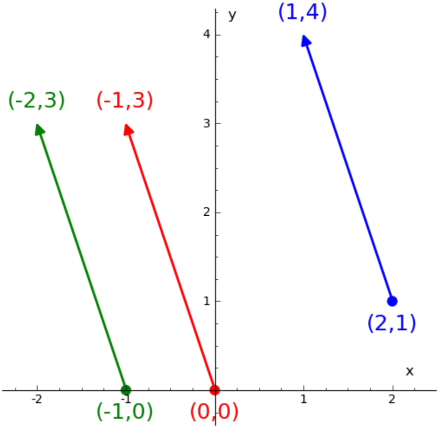
Blue vector:
Initial point $(x_i, y_i) = (2, 1)$
Terminal point $(x_t, y_t) = (1, 4)$
blue vector equals $(x_t - x_i, y_t - y_i) = (1 - 2, 4 - 1) = (-1, 3)$.

Calculating a Vector, continued



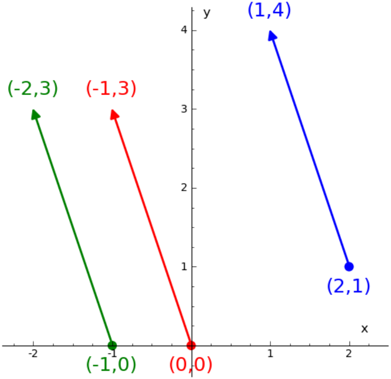
Red vector: Initial point $(x_i, y_i) = (0, 0)$
Terminal point $(x_t, y_t) = (-1, 3)$
red vector equals $(x_t - x_i, y_t - y_i) = (-1 - 0, 3 - 0) = (-1, 3)$.

Translation



Changing the initial and final points of a vector without changing the vector itself (i.e., it keeps the same length and direction) is called *translation*. The three versions of the same vector appearing above all called *translates* of each other.

Vectors Beginning at the Origin



If a vector (u, v) has initial point (x_i, y_i) equal to the origin $(0, 0)$, and terminal point (x_t, y_t), then

$$(u, v) = (x_t - x_i, y_t - y_i) = (x_t - 0, y_t - 0) = (x_t, y_t).$$

So a vector with initial point at the origin is given the same notation as its terminal point.

2-Dimensional Vectors, Algebraically

Algebraically: 2-dimensional vectors are represented by $\mathbb{R}^2$, the set of all ordered pairs (x, y) of real numbers

Recall that the set of points in two-dimensional space is also represented algebraically by $\mathbb{R}^2$.

So algebraically, vectors and points share the same representation: this is because of the correspondence we just showed between a point (x, y) and the vector from $(0, 0)$ to (x, y).

In the algebraic view, two vectors (a, b) and (c, d) are equal if their components match: that is $(a, b) = (c, d)$ if and only if we have both $a = c$ and $b = d$.

Vectors in Other Dimensions

Vectors also represent displacements in other dimensions.

In n-Dimensional Space, the vector u with initial point $(a_1, a_2, \ldots, a_n)$ and terminal point $(b_1, b_2, \ldots, b_n)$ is just

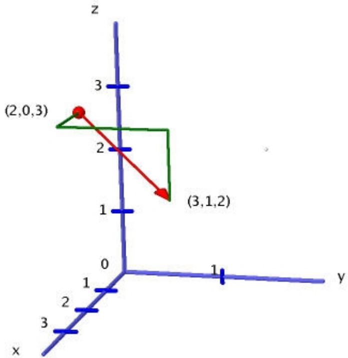
$$\mathbf{u} = (u_1, u_2, \ldots, u_n) = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n),$$

and $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is equal to $\mathbf{v} = (v_1, \ldots, v_n)$ if and only if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n.$$

Note that in texts, **boldface** is often used for a letter that represents an entire vector, while the components are not written in boldface.

A 3-Dimensional Vector



Initial point: $(2, 0, 3)$, Terminal Point: $(3, 1, 2)$
Vector is thus: $(3 - 2, 1 - 0, 2 - 3) = (1, 1, -1)$.

(Green lines show that we can get from the initial to terminal point by going 1 unit in the positive x-direction, 1 unit in the positive y-direction, and 1 unit in the negative z-direction.)

Other Notations for Vectors

The form $(x_1, x_2, \ldots, x_n)$ we have been using is called *comma-delimited form*.

There is also *row vector* form, written as

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}$$

or

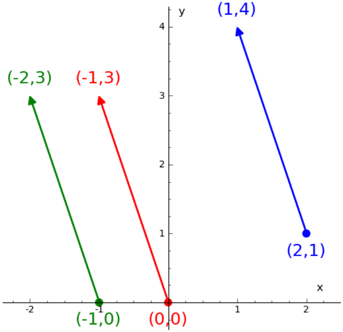
$$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix},$$

and *column vector* form, written as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Length of Vectors

Use the Pythagorean theorem to compute the length of vectors.



The length of the vector $(-1, 3)$ is $\sqrt{(-1)^2 + 3^2} = \sqrt{10}$.

More generally, if $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, then the length of $\mathbf{u}$, written $||\mathbf{u}||$, is $\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$.

The Zero Vector

In our algebraic view, there is a vector $(0, 0)$

This corresponds to zero displacement (initial point=terminal points)

Its length is of course $\sqrt{0^2 + 0^2} = 0$

Its direction is unclear: we can give it any direction we want.

Algebraic Vector Addition

We have an algebraic interpretation of vectors as elements of $\mathbb{R}^2$, that is, ordered pairs (x, y) of real numbers

So let's introduce an algebraic operation of addition.

The *sum* of two vectors (a, b) and (c, d) is obtained by summing component-wise:

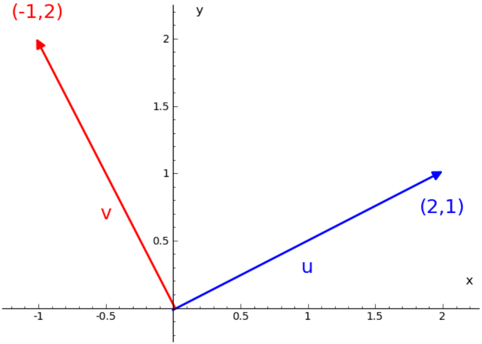
$$(a, b) + (c, d) = (a + c, b + d).$$

More generally, for n-dimensional vectors, we have

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

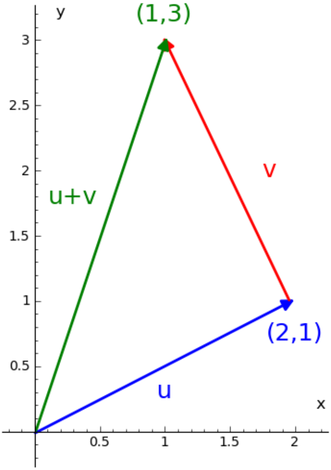
Vector Addition, Geometric Interpretation

Suppose we want to sum two vectors **u** and **v**.



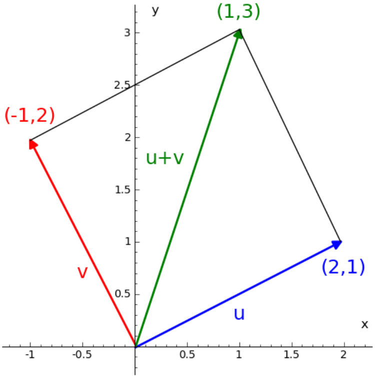
Vector Addition, Triangle Rule

To sum **u** and **v**, place **u**'s initial point at the origin, and translate **v** so that its initial point is at the terminal point **u**. Then $\mathbf{u} + \mathbf{v}$ is the vector from the origin to terminal point of the now-translated **v**.



Vector Addition, Parallelogram Rule

To sum **u** and **v**, place both **u**'s initial point and **v**'s initial point at the origin, and then complete the parallelogram that has these two vectors as its sides. Then $\mathbf{u} + \mathbf{v}$ is the vector from the origin to the opposite corner of the parallelogram.

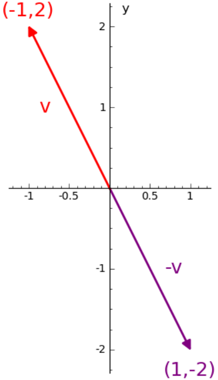


This shows that the order of addition doesn't matter:
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Negation of a Vector

The negative of a vector $\mathbf{v}$ is obtained algebraically by negating all the coordinates: $-(x, y) = (-x, -y)$ in two dimensions or $-(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, \ldots, -x_n)$ in n dimensions.

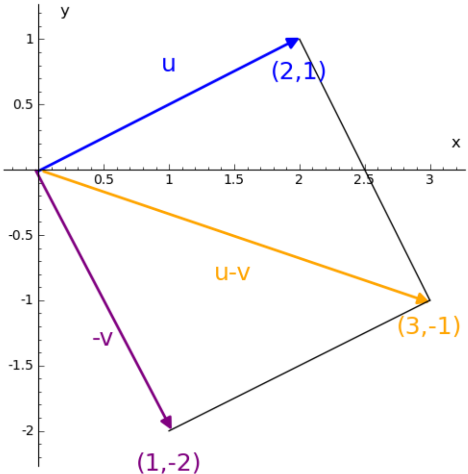
Geometrically, it is the vector with the same length but opposite direction.



Vector Subtraction

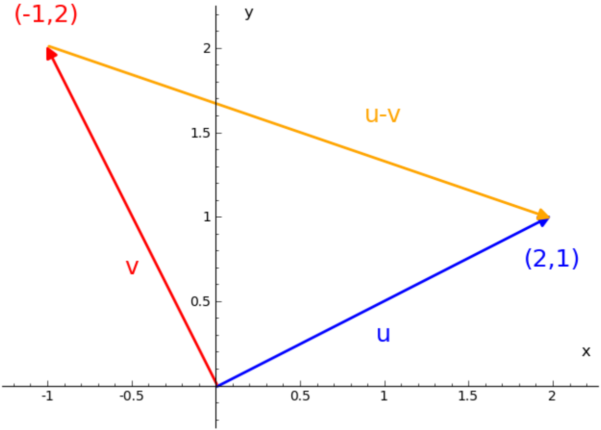
Algebraically $(a, b) - (c, d) = (a - c, b - d)$ or $(a_1, a_2, \ldots, a_n) - (b_1, b_2, \ldots, b_n) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$.

So the difference of vectors $\mathbf{u} - \mathbf{v}$ is just $\mathbf{u} + (-\mathbf{v})$. Geometrically, we use the parallelogram rule to add $\mathbf{u}$ to $-\mathbf{v}$.



Vector Subtraction

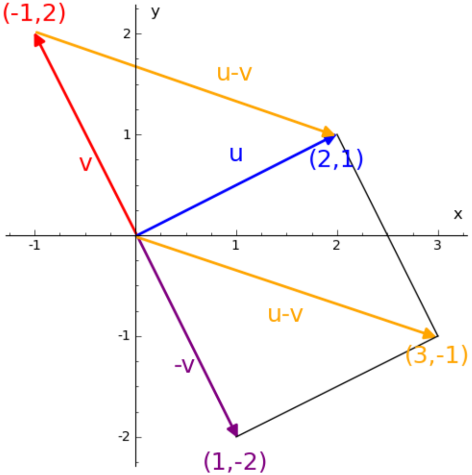
Another interpretation of $\mathbf{u} - \mathbf{v}$ is the vector we get if we go from the terminal point of $\mathbf{v}$ to the initial point of $\mathbf{u}$.



Order does matter!

Vector Subtraction

Both interpretations of subtraction give the same vector: they are just translates of each other.



Multiplication of a Vector by a Scalar, Algebraically

Algebraically if a is a scalar (number) and (x, y) or $(x_1, x_2, \ldots, x_n)$ is a vector, then

$$a(x, y) = (ax, ay)$$

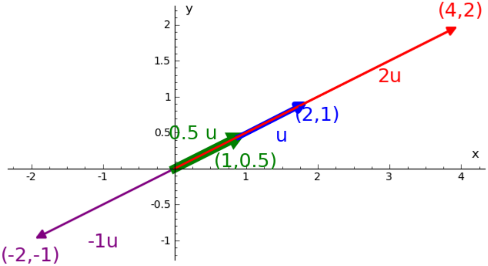
and

$$a(x_1, x_2, \ldots, x_n) = (ax_1, ax_2, \ldots, ax_n).$$

That is, you multiply each component by a.

Multiplication of a Vector by a Scalar, Geometrically

Geometrically, multiplication by $a > 0$ multiplies the length by a without changing the direction. If $a < 0$, then the length is scaled by $|a|$ and the direction reverses. If $a = 0$, then multiplication by a gives the zero vector.



Note that $-1\mathbf{u} = -\mathbf{u}$.

Parallel Vectors

We say that vectors $\mathbf{u}$ and $\mathbf{v}$ are *parallel* if one is a scalar multiple of the other, i.e., $\mathbf{u} = a\mathbf{v}$ for some number a, or if $\mathbf{v} = b\mathbf{u}$ for some number b.

In principle we have to check both, since $(0,0) = 0(1,2)$, but $(1,2) \neq a(0,0)$ for any number a.

If you already know that both vectors are nonzero, you only need to check one of the relations.

So $(0,0)$ is parallel to every vector. This is why we say that $(0,0)$ can have any direction we want.

The zero vector is written as **boldface** zero, $\mathbf{0}$.

Algebraic Rules for Vectors

Now that we have the basic operations (addition, negation, subtraction, scalar multiplication), we have algebraic rules that resemble those familiar for numbers, but note that scalars and vectors are different objects, so things are a bit more complicated. For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and scalars a and b, we have

$$\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} \\
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \mathbf{u}+(\mathbf{v}+\mathbf{w}) \\
\mathbf{u}+\mathbf{0} &= \mathbf{u} \\
\mathbf{u}+(-\mathbf{u}) &= \mathbf{0} \\
(a+b)\mathbf{u} &= a\mathbf{u}+b\mathbf{u} \\
a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v} \\
a(b\mathbf{u}) &= (ab)\mathbf{u} \\
1\mathbf{u} &= \mathbf{u} \\
(-1)\mathbf{u} &= -\mathbf{u} \\
0\mathbf{u} &= \mathbf{0} \\
a\mathbf{0} &= \mathbf{0}
\end{aligned}$$

Linear Combinations

If $\mathbf{u}$ and $\mathbf{v}$ are vectors, a *linear combination* of $\mathbf{u}$ and $\mathbf{v}$ is any vector of the form $a\mathbf{u} + b\mathbf{v}$ where a and b are scalars.

Thus if $\mathbf{u} = (4, 1)$ and $\mathbf{v} = (-2, 3)$, the following are just a few of the (infinitely many) linear combinations of $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned} 2\mathbf{u} + 3\mathbf{v} &= 2(4,1) + 3(-2,3) = (8,2) + (-6,9) = (2,11) \\ 1\mathbf{u} + (-2)\mathbf{v} &= 1(4,1) + (-2)(-2,3) = (4,1) + (4,-6) = (8,-5) \\ 0\mathbf{u} + (-1)\mathbf{v} &= 0(4,1) + (-1)(-2,3) = (0,0) + (2,-3) = (2,-3). \end{aligned}$$

You can also linearly combine larger collections of vectors: a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ is any vector of the form

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k,$$

where $a_1, a_2, \ldots, a_k$ are scalars.

Matrices

An $m \times n$ *matrix* is an array of numbers with m rows and n columns.

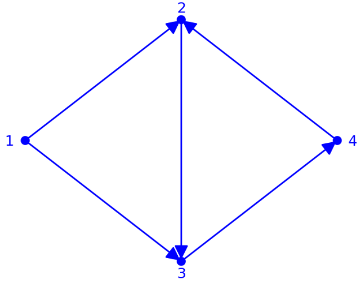
$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}.$$

A matrix with only one row is a row vector, and a matrix with only one column is a column vector.

Matrices are used to organize information.

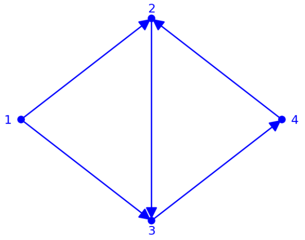
The Matrix of a Directed Graph

The following is a directed graph: which is a system of arrows connecting vertices.



It might model a transportation network (with vertices representing cities and arrows representing rail links) or a network of webpages (with vertices representing pages and arrows hyperlinks): note that the links are directed.

The Matrix of a Directed Graph, continued

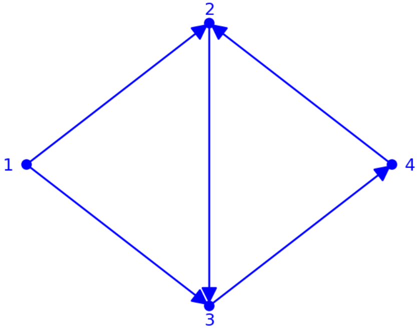


We can write a matrix, called the *connectivity matrix*, that describes this graph.

The rth row of the matrix will represent links departing from vertex r and the cth column will represent links entering vertex c. We put a 1 in the entry for row r and column c if there is a link from vertex r to vertex c; otherwise we put a 0.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The Matrix of a Directed Graph, concluded



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The 1 in row 2 and column 3 indicates that there is a link from vertex 2 going to vertex 3. The 0 in row 1 and column 4 indicates that there is no link from vertex 1 going to vertex 4.