Midterm 1
Math 184 Fall 2011

- No calculators, cell phones, formula sheets (etc) are permitted.
- This exam has 8 pages, including this cover page and the scrap paper page. Make sure you have all 8 pages.
- Show all your work - a correct answer will receive full marks only if it is fully justified.

LAST NAME: SOLUTIONS

FIRST NAME: ____________________

STUDENT NUMBER: ____________________

INSTRUCTOR: ____________________

Question	Marks
1	10
2	10
3	10
4	10
5	10
Total	50

1. (10 marks)

OR, $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

(a) Carefully state the definition of the derivative of a function $f(x)$ at a point $x = a$.

The derivative of $f(x)$ at $x=a$ is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, if this limit exists.

If this limit does not exist, then f is not differentiable at $x=a$.

(b) Use the definition of the derivative from part (a) to compute $f'(1)$ for $f(x) = \frac{7}{3x-1}$.

NO CREDIT WILL BE GIVEN FOR ANY OTHER METHOD

$$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \left[\left(\frac{7}{3(1+h)-1} - \frac{7}{3-1}\right)/h\right]$$

$$= \lim_{h \to 0} \left[\left(\frac{7}{2+3h} - \frac{7}{2}\right)/h\right]$$

$$= \lim_{h \to 0} \left[\left(\frac{7 \cdot 2 - 7 \cdot (2+3h)}{2 \cdot (2+3h)}\right)/h\right]$$

$$= \lim_{h \to 0} \left(\frac{-21h}{4+6h}\right)/h = \lim_{h \to 0} \frac{-21}{4+6h} = \frac{-21}{4+6 \cdot 0} = \frac{-21}{4}$$

(c) Find the equation of the tangent line to the graph of f at the point $(1, \frac{7}{2})$.

$$y = \frac{7}{2} - \frac{21}{4}(x-1) \qquad \left(\text{or, } y = \frac{35}{4} - \frac{21}{4}x\right)$$

(This is the point-slope formula for the line through $(1, \frac{7}{2})$ with slope m)

2. (10 marks) Compute the following limits, or explain why they do not exist.

(a) $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4}$ $= \lim_{x\to 4} \frac{(x)(x-4)}{(x+1)(x-4)}$

$= \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{5}$

(b) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x^2-81}$ $= \lim_{x\to 9} \left(\frac{\sqrt{x}-3}{(x-9)(x+9)} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$

$= \lim_{x\to 9} \frac{x-9}{(x-9)(x+9)(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{1}{(x+9)(\sqrt{x}+3)} = \frac{1}{18(3+3)} = \frac{1}{18\cdot 6}$

$\left(= \frac{1}{108}\right)$

(c) $\lim_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} e^x(x^2+2) & ,x \le 0 \\ \sqrt{x^{3/2}+1} & ,x > 0 \end{cases}$$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x^{3/2}+1} = \sqrt{0+1} = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x(x^2+2) = e^0(0+2) = 2$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist

3. (10 marks)

(a) Compute $f'(x)$ for $f(x) = \dfrac{x-1}{x+\sin(x)}$. DO NOT SIMPLIFY.

$$f'(x) = \frac{(1)(x+\sin x) - (x-1)(1+\cos x)}{(x+\sin x)^2}$$

(b) Compute $f'(x)$ for $f(x) = e^{7x^2}(x^2+1)$. DO NOT SIMPLIFY.

$$f'(x) = \left(e^{7x^2} \cdot 14x\right) \cdot (x^2+1) + e^{7x^2}(2x)$$

(c) If f and g are differentiable functions with $f(5) = 5$, $f'(5) = 0$, $g(5) = 3$, $g'(5) = -1$, $f'(3) = -2$ and $g'(3) = 2$, then compute $h'(5)$, where

$$h(x) = f(g(x)) + 4xf(x)$$

$$h'(x) = \left(f'(g(x)) \cdot g'(x)\right) + \left(4f(x) + 4x\,f'(x)\right)$$

$$h'(5) = f'(g(5)) \cdot g'(5) + 4f(5) + 4 \cdot 5 \; f'(5)$$

$$= f'(3) \cdot -1 + 4 \cdot 5 \quad 20 \cdot 0$$

$$= -2 \cdot -1 + 20 + 0 = 22$$

(d) Find the break-even points for Sprockets of Peace. Give both the price p and the quantity q at each of these points.

$R(q) = C(q)$

$-\frac{1}{50} q^2 + 200q = 20q$

$-q^2 + 10000q = 1000q$

$-q^2 + 9000q = 0$

$q(9000 - q) = 0$

$q = 0$ OR $q = 9000$

$\Rightarrow p = \frac{-1}{50} \cdot 0 + 200 = 200$

$\Rightarrow p = -\frac{9000}{50} + 200 = -180 + 200 = 20$

So $(0, 200)$ and $(9000, 20)$ are the breakeven points.

(e) If Entertainment Arbitrator is operating at the break-even point with highest q-value, should it increase or decrease the price to increase its profit? Use the marginal profit function in your explanation.

$P(q) = R(q) - C(q) = \frac{-1}{50} q^2 + 200q - 20q$

$= \frac{-1}{50} q^2 + 180q$

$mP(q) = \frac{-1}{25} q + 180$

$mP(9000) = \underbrace{\frac{-1}{25} \cdot 9000}_{= -180} + 180 < 0$

Since $mP(9000) < 0$, profit is increased by decreasing production.

This corresponds to increasing price

(f) How many copies of the game should Entertainment Arbitrator produce to maximize profit?

Solve $MP(q) = 0$ for q:

$\frac{-1}{25} q + 180 = 0$

$\frac{1}{25} q = 180$

$q = 25 \cdot 180$

They should sell $25 \cdot 180$ $(= 4500)$ copies to maximize profit

4. (10 marks) The publisher Entertainment Arbitrator sells the popular video game Sprockets of Peace. When each copy of the game is priced at \$40, the weekly demand is 8000 copies. For every \$1 increase in price, the number sold per week decreases by 50. Assume that it costs \$20 to produce each copy of the game.

(a) Find the linear demand equation. Use p for the unit price and q for the weekly demand.

(8000, 40) and (7950, 41) are on the line

slope $= \frac{\Delta p}{\Delta q} = \frac{-1}{50}$

$p = mq + b = \frac{-1}{50}q + b$

Plugin (8000, 40) : $40 = \frac{-1}{50}(8000) + b$

$\Rightarrow 40 = -160 + b$

$b = 200$

$\boxed{\text{So, } p = \frac{-1}{50}q + 200 \text{ is the demand equation, OR, } q = -50p + 1000}$

(b) Find the weekly cost function $C(q)$ as a function of q.

$C(q) = 20q$

(c) Find the weekly revenue function $R(q)$ as a function of q.

$R(q) = p \cdot q = \left(\frac{-1}{50}q + 200\right) q$

$= \frac{-1}{50}q^2 + 200q$

5. (10 marks) Show that the function $f(x) = e^x(2700x^8 + 1000x^6 - 1)$ touches the x-axis **at least TWICE** on the interval $[-1, 1]$. (Justify each step of your answer.)

$f(x)$ is the product of e^x and $2700x^8+1000x^6-1$, which are both continuous on $[-1,1]$. We know that the product of continuous functions is continuous. Thus, $f(x)$ is continuous on $[-1,1]$, and so the intermediate value theorem applies.

$f(-1) = e^{-1}(3699) > 0$

$f(0) = -1 < 0$

By the IVT, there exists c_1 in $(-1,0)$ with $f(c_1) = 0$

$f(0) = -1 < 0$

$f(1) = e(3699) > 0$

By the IVT, there exists c_2 in $(0,1)$ with $f(c_2) = 0$.

Since $(-1,0)$ and $(0,1)$ do not overlap, c_1 and c_2 are different, and both in the interval $[-1,1]$

Since "touching the x-axis" at a point a means $f(a)=0$, we have that $f(x)$ touches the x-axis twice in the interval $[-1,1]$ (at $x=c_1$ and at $x=c_2$)