Marks

[12] **1. Short Answer Questions.** Put your answer in the box provided. **Simplify your answer as much as possible.** Full marks will be awarded for a correct answer placed in the box, you do not need to show work. However, **for partial marks**, you should show your work. Each question is worth 3 points, but not all questions are of equal difficulty.

(a) Find a fundamental set of solutions of the following system of equations, and calculate the Wronskian.

$$\mathbf{x}'(t) = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \mathbf{x}(t)$$

Answer

By inspection the eigenvalues are 3 and 4 (real and distinct), with corresponding eigenvectors $(1, 0)$ and $(1, 1)$. A fundamental set is then $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}\}$. The Wronskian is

$$\begin{aligned} W &= \det \begin{pmatrix} e^{3t} & e^{4t} \\ 0 & e^{4t} \end{pmatrix} \\ &= e^{7t} \neq 0. \end{aligned}$$

(b) Determine whether the origin is a **saddle point**, **unstable node**, or **asymptotically stable node** of the following system of equations:

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & 0 \\ 3 & -6 \end{pmatrix} \mathbf{x}(t)$$

Answer

The eigenvalues are immediately seen to be -2 and -6, which are distinct, real eigenvalues and both negative. Therefore, the origin is an **asymptotically stable node**.

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(c) Suppose that the vibration of a mass on a spring is modeled by the equation

$$my'' + y' + 3y = 0.$$

Find all values of $m > 0$ for which this system is **underdamped**.

Answer

The system is underdamped when the roots of the characteristic polynomial are nonreal, i.e. when (by looking at the quadratic formula) we have $b^2 - 4ac = 1 - 12m < 0$ or equivalently

$$m > \frac{1}{12}$$

(the strict inequality is important here).

(d) Find the Laplace transform of the function g, where $g(t)$ is defined by

$$g(t) = \begin{cases} 0, & 0 \leq t < 6, \\ (t-4)^2, & 6 \leq t. \end{cases}$$

Answer

First re-write g using step functions: $g(t) = u_6(t)(t-4)^2$. We use the following formula from the table: $\mathcal{L}\{u_c(t)f(t-c)\}(s) = e^{-cs}\mathcal{L}\{f(t)\}(s)$. Note that $c = 6$, and $f(t) = t+2$ (then $f(t-6) = t - 6 + 2 = t - 4$ and the expressions match). This means

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= \mathcal{L}\{u_6(t)(t-4)^2\}(s) \\ &= e^{-6s}\mathcal{L}\{(t+2)^2\}(s) \\ &= e^{-6s}(\mathcal{L}\{t^2\}(s) + 4\mathcal{L}\{t\}(s) + 4\mathcal{L}\{1\}(s)) \\ &= e^{-6s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right). \end{aligned}$$

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Full-Solution Problems. In questions 2–5, justify your answers and **show all your work**. If a box is provided, write your final answer there.

[9] **2.** Using the Method of Undetermined Coefficients, find any particular solution of the following equation:

$$y'' - 2y' + y = 3e^t + e^{3t}.$$

Any solution that does not use the Method of Undetermined Coefficients will receive a mark of zero.

First, the characteristic polynomial of this equation is $r^2-2r+1 = (r-1)^2$ so $r = 1$ is a repeated root, so both e^t and te^t are solutions to the corresponding homogeneous equation. This means instead of the usual guess of Ae^t or Ate^t for the first term on the right hand side, we need to pick At^2e^t. Combined with the second term, this tells us our guess should be of the form: $Y(t) = At^2e^t + Be^{3t}$. We calculate dervatives and plug in: $Y' = 2Ate^t+At^2e^t+3Be^{3t}$, $Y'' = 2Ae^t+4Ate^t+At^2e^t+9Be^{3t}$, and

$$\begin{aligned} 3e^t + e^{3t} &= Y'' - 2Y' + Y \\ &= 2Ae^t + 4Ate^t + At^2e^t + 9Be^{3t} - 4Ate^t - 2At^2e^t - 6Be^{3t} + At^2e^t + Be^{3t} \\ &= 2Ae^t + 4Be^{3t}. \end{aligned}$$

Equating the coefficients, $A = 3/2$ and $B = 1/4$ so a particular solution can be

$$Y(t) = \frac{3}{2}t^2e^t + \frac{e^{3t}}{4}.$$

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[12] **3.** Find the general solution of the following equation using Variation of Parameters:

$$y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{\ln t}{t^3}.$$

You may use the fact that $\{\frac{1}{t}, \frac{1}{t^2}\}$ is a fundamental set of solutions of the homogeneous equation $y'' + \frac{4}{t}y' + \frac{2}{t^2}y = 0$.

Evaluate all integrals in your final answer.
Any solution that does not use Variation of Parameters will receive a mark of zero.

Divide the equation through by t^2 first to get $y'' + 4t^{-1}y' + 2t^{-2} = t^{-3}\ln t$. Now we set $y(t) = u_1(t)t^{-1} + u_2(t)t^{-2}$. Then the system of equations to solve is

$$u_1' t^{-1} + u_2' t^{-2} = 0$$
$$u_1'(-t^{-2}) + u_2'(-2t^{-3}) = t^{-3}\ln t.$$

By the first equation, $u_1' = -u_2' t^{-1}$, plugging into the second equation we get $u_2' t^{-3} - 2t^{-3}u_2' = t^{-3}\ln t$ or

$$u_2' = -\ln t$$
$$u_1' = \frac{\ln t}{t}.$$

Thus, $u_1 = \int \frac{\ln t}{t} dt = \frac{(\ln t)^2}{2} + c_1$ and $u_2 = -\int \ln t dt = -t\ln t + t + c_2$ (use substitution for the first, integration by parts for the second). Thus putting it all together:

$$y(t) = c_1 t^{-1} + c_2 t^{-2} + \frac{(\ln t)^2}{2t} - \frac{\ln t}{t}$$

(there is actually a t^{-1} term at the end, but it can be absorbed into the first $c_1 t^{-1}$ term).

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[12] **4.** Find the solution of the following initial value problem using the Laplace transform:

$$\frac{dy}{dt} = -y + 2u_3(t),\ y(0) = 4.$$

Here, $u_3(t)$ is the unit step function with step at $t = 3$.

Any solution that does not use the Laplace transform will receive a mark of zero.

Re-write the equation to $y' + y = 2u_3(t)$ and Laplace transform both sides:

$$\begin{aligned}
\mathcal{L}\{y'\}(s) + \mathcal{L}\{y\}(s) &= 2\mathcal{L}\{u_3(t)\}(s) \\
s\mathcal{L}\{y\}(s) - y(0) + \mathcal{L}\{y\}(s) &= 2\frac{e^{-3s}}{s} \\
\mathcal{L}\{y\}(s)(s+1) - 4 &= 2\frac{e^{-3s}}{s} \\
\mathcal{L}\{y\}(s) &= 2\frac{e^{-3s}}{s(s+1)} + \frac{4}{s+1}.
\end{aligned}$$

Then, using partial fractions,

$$\begin{aligned}
2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s(s+1)}\right\}(t) &= 2u_3(t)\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t-3) \\
&= 2u_3(t)\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t-3) \\
&= 2u_3(t)\left(1 - e^{-(t-3)}\right).
\end{aligned}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) = 4e^{-t}$ directly from the table ($a = -1$), we obtain

$$y(t) = 2u_3(t)\left(1 - e^{-(t-3)}\right) + 4e^{-t}.$$

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[5] **5.** Let $a > 0$ and f be a continuous function for which $\int_0^\infty f(x)dx$ is convergent, and define the function g for any $t > 0$ by

$$g(t) = \int_0^{at} f(x)dx.$$

Also suppose that $\mathcal{L}\{f(t)\}(s)$ exists for all $s > 0$.

Then using the definition of the Laplace transform, show that for any $s > 0$ we have

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(\frac{s}{a}).$$

(Hint: You may want to use integration by parts).

First we apply integration by parts (use the Chain Rule and Fundamental Theorem of Calculus to get du) with

$$u = \int_0^{at} f(x)dx, \qquad dv = e^{-st}dt$$

$$du = af(at)dt, \qquad v = -\frac{e^{-st}}{s}$$

to get

$$\begin{aligned}\mathcal{L}\{g(t)\}(s) &= \lim_{A\to\infty}\int_0^A \left(e^{-st}\int_0^{at} f(x)dx\right)dt \\ &= \lim_{A\to\infty}\left(-\frac{e^{-st}}{s}\int_0^{at} f(x)dx\Big]_{t=0}^{A} + \frac{a}{s}\int_0^A e^{-st}f(at)dt\right).\end{aligned}$$

By the assumptions on f and since a, $s > 0$, we have $\lim_{A\to\infty}\left(-\frac{e^{-sA}}{s}\int_0^{aA} f(x)dx\right) = 0$, while the expression is clearly zero when $t = 0$. By applying a substitution, (and since $a > 0$) we then see

$$\begin{aligned}\lim_{A\to\infty}\frac{a}{s}\int_0^A e^{-st}f(at)dt &= \frac{1}{s}\lim_{A\to\infty}\int_0^{aA} e^{-(s/a)t}f(t)dt \\ &= \frac{1}{s}\int_0^\infty e^{-(s/a)t}f(t)dt \\ &= \frac{1}{s}\mathcal{L}\{f(t)\}(\frac{s}{a})\end{aligned}$$

as desired.

Alternatively, you can say for the last step, using the table

$$\begin{aligned}\lim_{A\to\infty}\frac{a}{s}\int_0^A e^{-st}f(at)dt &= \frac{a}{s}\int_0^\infty e^{-st}f(at)dt \\ &= \frac{a}{s}\mathcal{L}\{f(at)\}(s) \\ &= \frac{a}{s}\cdot\frac{1}{a}\mathcal{L}\{f(t)\}(\frac{s}{a}) \\ &= \frac{1}{s}\mathcal{L}\{f(t)\}(\frac{s}{a})\end{aligned}$$

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The University of British Columbia
Sessional Examinations - Mar. 15, 2013

Mathematics 215
Elementary Differential Equations I

Closed book examination Time: 50 minutes

Last Name: ____________________ **First Name:** ____________________

Student Number:____________________ **Instructor's Name:** ____________________

Signature:____________________ **Section Number:** ____________________

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1		12
2		9
3		12
4		12
5		5
Total		50