Marks

[12] **1. Short Answer Questions.** Put your answer in the box provided. **Simplify your answer as much as possible.** Full marks will be awarded for a correct answer placed in the box. **For partial marks**, show your work. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find a fundamental set of solutions of the following equation, and show that the Wronskian is not equal to zero.

$$y'' + 2y' - 3y = 0.$$

Answer

The characteristic polynomial is $r^2 + 2r - 3 = (r-1)(r+3)$, so a fundamental set is $\{e^t,\ e^{-3t}\}$. The Wronskian is

$$\begin{aligned} W(e^t, e^{-3t}) &= \det \begin{pmatrix} e^t & e^{-3t} \\ e^t & -3e^{3t} \end{pmatrix} \\ &= -3e^t e^{-3t} - e^t e^{-3t} \\ &= -4e^{-2t} \neq 0. \end{aligned}$$

Other choices are possible.

(b) Write down an integrating factor for the equation

$$3ty' + (t\cos t)y = t^t,\ t > 0$$

(you do not need to solve the equation).

Answer

Again, divide the equation by $3t$ to get

$$y' + \frac{\cos t}{3} y = \frac{t^t}{3t}.$$

Since $p(t) = \dfrac{\cos t}{3}$, an integrating factor is $\mu(t) = e^{\int (\cos t/3) dt} = e^{(\sin t/3)}$.

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(c) Suppose that $y(t)$ is a solution to the initial value problem

$$(t-2)y'' + (\ln t)y = 0,\ y(1) = 1,\ y'(1) = 2.$$

What is the largest open interval on which $y(t)$ is **guaranteed** to exist?

Answer

First divide the equation by $(t-2)$ to get

$$y'' + \frac{\ln t}{t-2}y = 0.$$

$\frac{\ln t}{t-2}$ is continuous on $(0,2) \cup (2,\infty)$, but our initial condition is given at $t = 1$ so we pick the biggest interval that contains $t = 1$. By the existence theorem for linear equations we are guaranteed existence on $(0,2)$.

(d) Write down the general solution of the equation

$$y'' - 2y' + 2y = 0.$$

Answer

The characteristic polynomial is $r^2 - 2r + 2$, and the roots are $r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ so a fundamental set is $\{e^t \cos t,\ e^t \sin t\}$ (we want real-valued solutions here). The general solution is

$$y(t) = c_1 e^t \cos t + c_2 e^t \sin t.$$

Other choices of the fundamental set are possible, but they should all end up the same after accounting for constant multiples.

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Full-Solution Problems. In questions 2–5, justify your answers and **show all your work**. If a box is provided, write your final answer there.

[8] **2.** Find the general solution to the following linear equation:

$$t^2\frac{dy}{dt} + ty = t^3,\ t > 0.$$

Divide both sides by t^2 to get

$$\frac{dy}{dt} + \frac{1}{t}y = t.$$

Then an integrating factor is $\mu(t) = e^{\int (1/t)dt} = e^{\ln t} = t$, multiply through to solve:

$$\begin{aligned}\frac{d}{dt}(ty) &= t^2\\ ty &= \int t^2 dt = \frac{t^3}{3} + C\\ y(t) &= \frac{C}{t} + \frac{t^2}{3}.\end{aligned}$$

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[9] **3.** Find the general solution for the following exact equation:

$$(xy^2 + 3x^2y) + (x^3 + x^2y)\frac{dy}{dx} = 0.$$

Note $M(x, y) = (xy^2 + 3x^2y)$, $N(x, y) = (x^3 + x^2y)$, and $M_y = 2xy + 3x^2 = N_x$, so this is an exact equation (you did not have to check this). First integrate M in the x variable:

$$\psi(x, y) = \int (xy^2 + 3x^2y)dx + h(y)$$
$$= \frac{x^2y^2}{2} + x^3y + h(y).$$

Then differentiate, and set equal to N to find $h(y)$:

$$\frac{\partial}{\partial y}\psi(x, y) = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{2} + x^3y\right) + \frac{dh}{dy}$$
$$= x^2y + x^3 + \frac{dh}{dy}$$
$$= N(x, y) = x^3 + x^2y$$

or

$$\frac{dh}{dy} = 0.$$

This means h is a constant, so $\psi(x, y) = \dfrac{x^2y^2}{2} + x^3y+$ a constant, and the implicit form of the solution is then

$$\frac{x^2y^2}{2} + x^3y = C.$$

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[5] **4.** The following equation is not exact:

$$(x+y)+(x\ln x-2xy)\frac{dy}{dx}=0,\ x>0.$$

Find an integrating factor that will turn this equation into an exact one (you **do not** have to solve the equation).

Note $M(x,y)=(x+y)$, $N(x,y)=(x\ln x-2xy)$. Since $M_y=1$ does not equal $N_x=\ln x+1-2y$, this equation is not exact (you did not have to check this). We assume our integrating factor is a function $\mu(x)$ of just the x variable. We want

$$\begin{aligned}(\mu M)_y &= (\mu N)_x\\ \mu M_y &= \mu' N+\mu N_x\\ \mu' &= \frac{M_y-N_x}{N}\mu\end{aligned}$$

(here $\mu'=d\mu/dx$). Since

$$\frac{M_y-N_x}{N}=\frac{1-(\ln x+1-2y)}{x\ln x-2xy}=-\frac{\ln x-2y}{x(\ln x-2y)}=-\frac{1}{x}$$

is a function of just x, we can solve the equation for μ:

$$\begin{aligned}\frac{\mu'}{\mu}&=-\frac{1}{x}\\ \ln\mu&=-\ln x=\ln\left(\frac{1}{x}\right)\end{aligned}$$

and so

$$\mu(x)=\frac{1}{x}$$

is an integrating factor.

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For this problem, we consider the following differential equation, where $\alpha > 1$ is some fixed constant:

$$\frac{dy}{dt} = \frac{(y-1)(y-\alpha)}{1-\alpha},\ t > 0.$$

[16] **5.** (a) [2] Find all equilibrium solutions of this equation.

Answer

Set the RHS equal to zero to get:

$$y = 1,\ \alpha.$$

(b) [3] Find all ranges of y on which the solutions of this equation are **strictly increasing**.

Since $\alpha > 1$, $(1-\alpha) < 0$. Then, we want $y' = \dfrac{(y-1)(y-\alpha)}{1-\alpha} > 0$, which is when $(y-1)(y-\alpha) < 0$, which in turn is when $1 < y < \alpha$.

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(c) [8] Write down, **in explicit form**, the solution of the initial value problem

$$\frac{dy}{dt} = \frac{(y-1)(y-\alpha)}{1-\alpha}, \ t > 0, \ y(0) = 2\alpha.$$

Recall, $\alpha > 1$ is a fixed constant.

This equation is separable so we integrate (and using partial fractions):

$$\begin{aligned} \int \frac{(1-\alpha)dy}{(y-1)(y-\alpha)} &= \int dt \\ \int \left(\frac{1}{y-1} - \frac{1}{y-\alpha} \right) dy &= t + C \\ \ln|y-1| - \ln|y-\alpha| &= t + C \\ \left| \frac{y-1}{y-\alpha} \right| &= Ce^t. \end{aligned}$$

Since $y(0) = 2\alpha > \alpha > 1$, both numerator and denominator of the LHS above will remain positive, so $\left| \frac{y-1}{y-\alpha} \right| = \frac{y-1}{y-\alpha}$. Then we have

$$\begin{aligned} \frac{y-1}{y-\alpha} &= Ce^t \\ y - 1 &= Ce^t y - C\alpha e^t \\ y(t) &= \frac{1 - C\alpha e^t}{1 - Ce^t}. \end{aligned}$$

Plugging in the initial condition, $C = \frac{2\alpha - 1}{\alpha}$, so

$$y(t) = \frac{1 - (2\alpha - 1)e^t}{1 - \frac{2\alpha - 1}{\alpha} e^t} = \frac{\alpha(1 - (2\alpha - 1)e^t)}{\alpha - (2\alpha - 1)e^t}.$$

(d) [3] Find $\lim_{\alpha \to 1^+} y_\alpha(t)$, where y_α is the solution to the initial value problem from part (c).
By the above,

$$\begin{aligned} \lim_{\alpha \to 1^+} y_\alpha(t) &= \lim_{\alpha \to 1^+} \left(\frac{1 - (2\alpha - 1)e^t}{1 - \frac{2\alpha - 1}{\alpha} e^t} \right) \\ &= \frac{1 - e^t}{1 - e^t} \\ &= 1 \end{aligned}$$

for all t. The key here is that the C in the solution above depended on α, so you had to figure out what it was **before** taking the limit.

The End