Solutions to 2012 ENGR 233/4 Final

#1 (a) $P_1\ (2,1,0)$ $P_2\ (3,4,0)$ $P_3\ (1,1,1)$

$\overrightarrow{P_1P_2} = (1,3,0)$ $\overrightarrow{P_3P_1} = (1,0,-1)$

$$\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_3P_1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3\hat{i} + \hat{j} - 3\hat{k}$$

$$-3x + y - 3z = -3(2) + 1(1) - 3(0)$$

$$-3x + y - 3z = -6 + 1 = -5$$

$$\boxed{3x - y + 3z = 5}$$

(b) $\vec{r}(t) = (3,4,0) + t(-3,1,-3)$

$$\boxed{\begin{aligned} x &= 3 - 3t \\ y &= 4 + t \\ z &= 0 - 3t \end{aligned}}$$

#2 (a) $w = \sqrt{x^3 + y} + e^{xz}$, $x = 2t$, $y = t^2$, $z = t^{-1}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{\partial w}{\partial x}(2) + \frac{\partial w}{\partial y}(2t) + \frac{\partial w}{\partial z}(-t^{-2})$$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^3+y)^{-1/2}(3x^2) + ze^{xz}$$

at $t = 1$: $x = 2$ $y = 1$ $z = 1$

$$\frac{\partial w}{\partial x}(2,1,1) = \frac{1}{2}(8+1)^{-1/2}(3 \cdot 4) + e^2 = \frac{1}{2} \cdot \frac{1}{3} \cdot 3 \cdot 4 + e^2 = (2 + e^2)$$

$$\frac{\partial w}{\partial y} = \frac{1}{2}(x^3+y)^{-1/2}$$

$$\frac{\partial w}{\partial y}(2,1,1) = \frac{1}{2}(8+1)^{-1/2} = \frac{1}{6}$$

$$\frac{\partial w}{\partial z} = xe^{xz}$$

$$\boxed{\frac{\partial w}{\partial z}(2,1,1) = 2e^2}$$

$$\frac{dw}{dt} = (2+e^2)2 + \frac{1}{6}(2 \cdot 1) + 2e^2(-1^{-2}) = 4 + 2e^2 + \frac{1}{3} - 2e^2 = \boxed{\frac{13}{3}}$$

#2(b) direction of fastest cooling: $-\vec{\nabla}T(2,3)$ (or unit vector)

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} = (10x+ye^{xy})\hat{i} + xe^{xy}\hat{j}$$

$$-\vec{\nabla}T(2,3) = -(20+3e^{6})\hat{i} - 2e^{6}\hat{j}$$

$$\hat{u} = \frac{-\vec{\nabla}T(2,3)}{\|\vec{\nabla}T\|} = \boxed{\frac{-(20+3e^{6})\hat{i} - 2e^{6}\hat{j}}{\sqrt{(20+3e^{6})^2+(2e^{6})^2}}}$$

directional derivative $D_{\hat{u}}T = \vec{\nabla}T\cdot\hat{u} = \frac{-\|\vec{\nabla}T\|^2}{\|\vec{\nabla}T\|}$

$$= -\|\vec{\nabla}T\| = -\sqrt{400+120e^{6}+9e^{12}+4e^{12}}$$

$$= \boxed{-\sqrt{400+120e^{6}+13e^{12}}}$$

#3. $g(x,y,z) = x^2+y^2+z = 10 =$ constant

$\vec{\nabla}g = 2x\hat{i}+2y\hat{j}+\hat{k}$ is $\perp$ to S

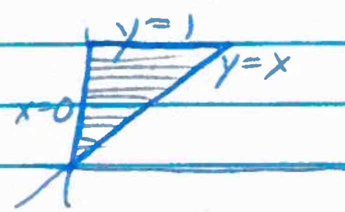
$$(2x, 2y, 1) = c\left(1, \tfrac{3}{2}, \tfrac{1}{2}\right) \Rightarrow c=2$$

$2x=2$, $x=1$ $\quad$ $2y=3$, $y=\frac{3}{2}$ $\quad$ $z = 10-x^2-y^2 = 10-1-\frac{9}{4}$

$$= 9-\frac{9}{4} = 9\cdot\frac{3}{4} = \frac{27}{4}$$

$$\boxed{\left(1, \tfrac{3}{2}, \tfrac{27}{4}\right)}$$

#4. $I_y = \iint_R x^2\rho(x,y)\,dxy$



$$= \int_0^1\int_0^y x^2\sqrt{1+y^4}\,dx\,dy$$

$$= \int_0^1 \frac{y^3}{3}\sqrt{1+y^4}\,dy = \frac{1}{3}\cdot\frac{2}{3}(1+y^4)^{3/2}\cdot\frac{1}{4}\Big|_0^1 = \frac{1}{18}\left(5^{3/2}-1^{3/2}\right)$$

$$= \boxed{\frac{\sqrt{125}-1}{18}}$$

#5 (a) $\text{grad } f = \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$

$= 2x e^{x^2} \cos z \, \hat{\imath} + z^4 \cos y \, \hat{\jmath} + \left(e^{x^2}(-\sin z) + 4z^3 \sin y\right)\hat{k}$

(b) $\text{div}(\text{grad } f) = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$= \frac{\partial}{\partial x}\left(2x e^{x^2} \cos z\right) + \frac{\partial}{\partial y}\left(z^4 \cos y\right) + \frac{\partial}{\partial z}\left(-e^{x^2} \sin z + 4z^3 \sin y\right)$

$= 2e^{x^2} \cos z + 4x^2 e^{x^2} \cos z - z^4 \sin y - e^{x^2} \cos z + 12z^2 \sin y$

(c) div f not possible since f is scalar, not vector

(d) $\text{curl}(\text{grad } f) = \vec{\nabla} \times \vec{\nabla} f = 0$ by identity proved in class

(e) grad (grad f)
not possible since grad f is a vector and we cannot apply grad to a vector field, only to scalar functions

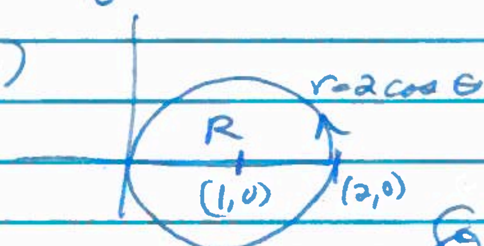
#6 (a) $\int_C -y\,dx + x\,dy$ $\quad P = -y \quad Q = x \quad \frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$

$x = 3\cos t$
$y = 2\sin t \qquad 0 \le t \le \pi$

$\int_0^\pi -2\sin t\,(-3\sin t\,dt) + 3\cos t\,(2\cos t\,dt)$

$= \int_0^\pi 6(\sin^2 t + \cos^2 t)\,dt = 6t\Big|_0^\pi = \boxed{6\pi}$

(b)



Green's Thm

$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\oint_C -y\,dx + x\,dy = \iint_R 1 - (-1)\,dA$

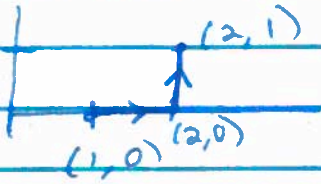
$= 2\iint_R 1\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta$

$= 2(\text{area of } R) = 2(\pi \cdot 1^2) = \boxed{2\pi}$

#7. $P = 1+e^{-y}$, $Q = -(xe^{-y}+4y)$

$\frac{\partial Q}{\partial x} = -e^{-y} = \frac{\partial P}{\partial y}$ and P, Q smooth in the whole plane $\Rightarrow$ indep. of path

$$\int_{(1,0)}^{(2,1)} P\,dx+Q\,dy = \int_1^2 (1+e^{-y})\,dx \;(y=0) + \int_0^1 -(xe^{-y}+4y)\,dy \;(x=2)$$



$$= 2\int_1^2 dx - \int_0^1 (2e^{-y}+4y)\,dy$$

$$= 2x\Big|_1^2 - (-2e^{-y}+2y^2)\Big|_0^1 = 2-(-2e^{-1}+2+2)$$

$$= \boxed{-2+2e^{-1}}$$

(using potential: $\phi(x,y) = \int(1+e^{-y})\,dx = x+xe^{-y}+g(y)$

$\frac{\partial \phi}{\partial y} = -xe^{-y}+g'(y) = Q = -xe^{-y}-4y \Rightarrow g'(y) = -4y$

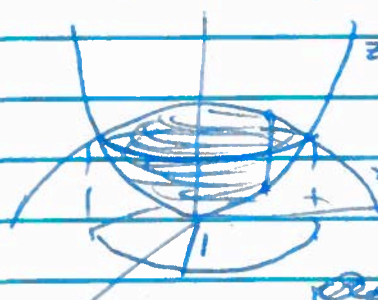
$\Rightarrow g(y) = -2y^2+C$

$\phi(x,y) = x+xe^{-y}-2y^2+C$

$$\int_C P\,dx+Q\,dy = \phi(x,y)\Big|_{(1,0)}^{(2,1)} = \phi(2,1)-\phi(1,0) = 2+2e^{-1}-2-(1+1e^0-0)$$

$$= \boxed{2e^{-1}-2}\;)$$

#8. Divergence Thm: $\oiint \vec{F}\cdot\hat{n}\,dS = \iiint_D (\vec{\nabla}\cdot\vec{F})\,dV$



$z = x^2+y^2 = r^2$

$x^2+y^2+z^2=2$

$r^2+z^2=2$

$$= \iiint \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 \left(\sqrt{2-r^2}-r^2\right) r\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\left[-\frac{1}{3}(2-r^2)^{3/2}-\frac{r^4}{4}\right]_0^1 d\theta$$

$$= 3\left(-\frac{1}{3}-\frac{1}{4}+\frac{1}{3}2^{3/2}\right)2\pi = \boxed{6\pi\left(\frac{\sqrt{8}-1}{3}-\frac{1}{4}\right)}$$

$r^2 = \sqrt{2-r^2}$

$r^4 = 2-r^2$

$r^4+r^2-2=0$

$r^2 = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = -2$ or 1

$\boxed{r=1}$

Stokes Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$ (5)

#10 (a)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

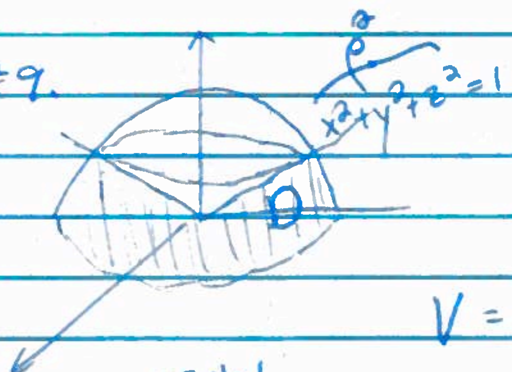
$$= \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz)}{\partial z}\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)\hat{k}$$

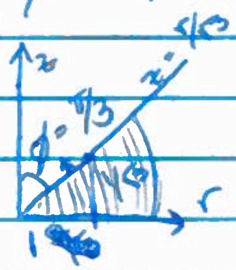
$$= (x-x)\hat{\imath} - (y-y)\hat{\jmath} + (z-z)\hat{k} = \vec{0}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS = \boxed{0}$$

(b) The answer is the same (0) by independence of path.

#9.



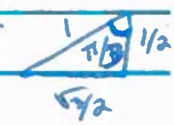


$3z^2 = x^2+y^2$

$3z^2 = r^2$

$z = \frac{r}{\sqrt{3}}$

$\tan\phi = \frac{r}{z} = \sqrt{3}$



$\phi = \pi/3$

$$V = \iiint_D 1\, dV$$

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \frac{\rho^3}{3}\Big|_0^1 \sin\phi\, d\phi\, d\theta = \frac{1}{3}\int_0^{2\pi} (-\cos\phi)\Big|_{\pi/3}^{\pi/2} d\theta = \frac{1}{3}\left(-0+\frac{1}{2}\right)2\pi$$

$$= \boxed{\frac{\pi}{3}}$$