

Question 1

(a) (i)

$$\lim_{x \rightarrow -3} \frac{x^2 - 3x + 2}{x - 1} = \frac{(-3)^2 - 3(-3) + 2}{-3 - 1} = -\frac{20}{4} = -5$$

(ii)

$$\begin{aligned} \lim_{x \rightarrow 5^+} \frac{x^2 - 16}{x - 5} &= \infty \\ \lim_{x \rightarrow 5^-} \frac{x^2 - 16}{x - 5} &= -\infty \end{aligned}$$

Thus, two-sided limit does not exist.

(iii)

$$\lim_{x \rightarrow \infty} \frac{-5x^7 + 3x^2 + 2}{4 - x^2} = \lim_{x \rightarrow \infty} \frac{-\frac{5x^7}{x^7} + \frac{3x^2}{x^7} + \frac{2}{x^7}}{\frac{4}{x^7} - \frac{x^2}{x^7}} = \lim_{x \rightarrow \infty} \frac{-5 + \frac{3}{x^5} + \frac{2}{x^7}}{\frac{4}{x^7} - \frac{1}{x^5}} = \infty$$

(b) (i)

$$\lim_{x \rightarrow 3} -3g(x) = -3 \lim_{x \rightarrow 3} g(x) = (-3)4 = -12$$

(ii)

$$\lim_{x \rightarrow 3} \sqrt{g(x)} = \sqrt{\lim_{x \rightarrow 3} g(x)} = \sqrt{4} = 2$$

(iii)

$$\lim_{x \rightarrow 3} \frac{g(x)}{2f(x)} = \frac{\lim_{x \rightarrow 3} g(x)}{2 \lim_{x \rightarrow 3} f(x)} = \frac{4}{-10} = -\frac{2}{5}$$

(iv)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(x-h)^2 - x^2}{h} &= \lim_{h \rightarrow 0} \frac{x^2 - 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-2x + h)}{h} \\ &= \lim_{h \rightarrow 0} (-2x + h) \\ &= -2x \end{aligned}$$

Question 2

(a)

$$f'(x) = (-6)10x^9 + (-4)3x^2 = -60x^9 - 12x^2$$

(b)

$$\begin{aligned} f'(x) &= \frac{(2x+5)(e^x-7) - e^x(x^2+5x)}{(e^x-7)^2} \\ &= \frac{2xe^x - 14x + 5e^x - 35 - e^xx^2 - 5xe^x}{(e^x-7)^2} \\ &= \frac{-x^2e^x - 3xe^x - 14x + 5e^x - 35}{(e^x-7)^2} \end{aligned}$$

(c)

$$\frac{dy}{dx} = \frac{2(x^2+3x)(2x+3)}{(x^2+3x)^2} = \frac{4x+6}{x^2+3x}$$

(d)

$$\frac{dy}{dx} = \frac{1}{5}(x+5)^{1/5-1} = \frac{1}{5}(x+5)^{-4/5} = \frac{1}{5\sqrt[5]{(x+5)^4}}$$

(e) Differentiate both sides of the equation with respect to x and use the Chain rule and the Product rule where appropriate:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(e^y - 2)$$

$$\frac{d}{dx}(x)y + x\frac{dy}{dx} = e^y\frac{dy}{dx}$$

$$\frac{dy}{dx}(e^y - x) = y$$

$$\frac{dy}{dx} = \frac{y}{e^y - x}$$

Question 3

The price-demand equation is:

$$x = f(p) = 7000 - 500p$$

Find the price elasticity of demand:

$$\begin{aligned} E(p) &= -\frac{pf'(p)}{f(p)} \\ &= \frac{500p}{7000 - 500p} E(p) \\ &= \frac{500p}{7000 - 500p} \leq 1 \end{aligned}$$

Hence,

$$\begin{aligned} 500p &\leq 7000 - 500p \\ p &\leq 7 \end{aligned}$$

Therefore the demand will remain inelastic while the price for a pair of sunglasses is less or equal to 7. Thus an increase in price will lead to an increase in revenue.

Question 4

Let $f(x) = x^2 - x$ and $g(x) = 2x$.

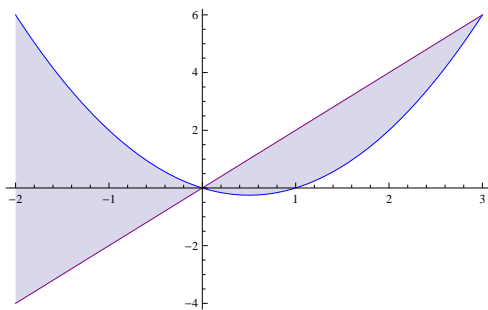


Figure 1: Plot of the parabola $x^2 - x$ and the line $2x$

First find the points of intersection of the two graphs:

$$\begin{aligned} x^2 - x &= 2x \\ x^2 - 3x &= 0 \\ x = 0, \quad x &= 3 \end{aligned}$$

On $[-2, 0]$, $f(x) > g(x)$. On $[0, 3]$, $g(x) > f(x)$. The area bounded by $f(x)$ and $g(x)$ is:

$$\begin{aligned}\int_{-2}^0 [f(x) - g(x)] dx + \int_0^3 [g(x) - f(x)] dx &= \int_{-2}^0 (x^2 - 3x) dx + \int_0^3 (3x - x^2) dx \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_{-2}^0 + \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 \\ &= \frac{8}{3} + \frac{12}{2} + \frac{27}{2} - \frac{27}{3} \\ &= \frac{79}{6}\end{aligned}$$

Question 5

$$f(x) = x^4 - 2x^3$$

Find the x -intercepts:

$$\begin{aligned}x^3(x - 2) &= 0 \\ x = 0, \quad x &= 2\end{aligned}$$

Find the critical points:

$$\begin{aligned}f'(x) &= 4x^3 - 6x^2 = 0 \\ 2x^2(2x - 3) &= 0 \\ x = 0, \quad x &= \frac{3}{2}\end{aligned}$$

Find the points of inflection:

$$\begin{aligned}f''(x) &= 12x^2 - 12x = 0 \\ 12x(x - 1) &= 0 \\ x = 0, \quad x &= 1\end{aligned}$$

	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 3/2)$	$3/2$	$(3/2, \infty)$
$f'(x)$	< 0	0	< 0	< 0	< 0	0	> 0
$f''(x)$	> 0	0	< 0	0	> 0	> 0	> 0
$f(x)$	Decreases, concave upward	Point of inflection	Decreases, concave downward	Point of inflection	Decreases, concave upward	Absolute minimum	Increases, concave downward

The graph of the function is:

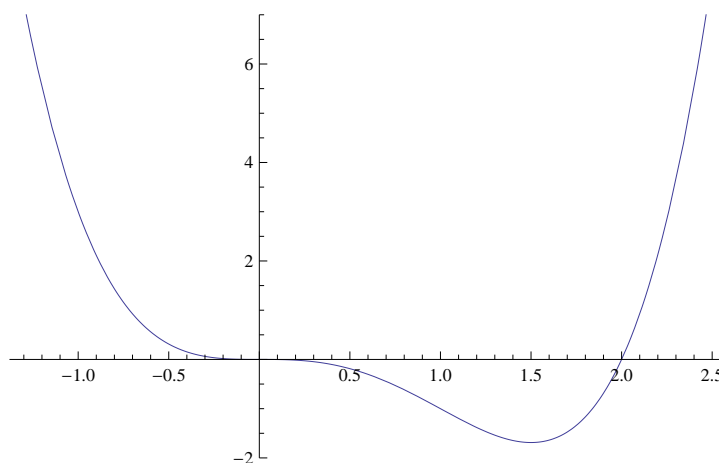


Figure 2: Plot of $x^4 - 2x^3$

Question 6

(a)

$$p = \frac{1000 - x}{20} = 50 - \frac{x}{20}, \quad 0 < x < 1000$$

(b)

$$R(x) = xp(x) = 50x - \frac{x^2}{20}, \quad 0 < x < 1000$$

(c) The marginal revenue at a production level of 400 units is:

$$R'(x) = 50 - \frac{x}{10}$$

$$R'(400) = 50 - 40 = 10$$

Therefore revenue is increasing at a rate of \$10 per unit.

(d) The marginal revenue at a production level of 650 units is:

$$R'(650) = 50 - 65 = -15$$

Therefore revenue is decreasing at a rate of \$15 per unit.

Question 7

$P = 500$ pounds/sq. inch, $T = 250$ Kelvin, $dT/dt = 3$ Kelvin/hour

$$k = \frac{P}{T} = \frac{500}{250} = 2$$

Since $P = 2T$, the rate of change of pressure with respect to time is:

$$\frac{dP}{dt} = 2 \frac{dT}{dt}$$

$$\frac{dP}{dt} = 2 \cdot 3 = 6 \text{ pounds}/(\text{sq. inch} \cdot \text{hour})$$

Question 8

(a)

$$\int e^{5x} dx = \frac{1}{5} e^{5x} + C$$

(b) Let $u = x - 7$. Then $du = dx$ and $x = u + 7$.

$$\begin{aligned} \int \frac{x}{\sqrt{x-7}} dx &= \int \frac{u+7}{\sqrt{u}} du \\ &= \int \left(\sqrt{u} + \frac{7}{\sqrt{u}} \right) du \\ &= \frac{2u^{3/2}}{3} + 14 u^{1/2} \\ &= \frac{2}{3}(x-7)^{3/2} + 14(x-7)^{1/2} + C \\ &= \sqrt{x-7} \left(\frac{2}{3}x - \frac{14}{3} + 14 \right) + C \\ &= \frac{2}{3} \sqrt{x-7}(x+14) + C \end{aligned}$$

(c)

$$\int (3x^2 + 5x) dx = x^3 + \frac{5}{2}x^2 + C$$

(d) Let $u = 4 + x^3$. Then $du = 3x^2$ or $1/3 du = x^2 dx$.

$$\int \frac{x^2}{4 + x^3} dx = \int \frac{1}{3u} du = \frac{1}{3} \ln |u| = \frac{1}{3} \ln |4 + x^3| + C$$

(e) Let $u = x^2 + 1$. Then $du = 2x dx$ or $1/2 du = x dx$.

$$\int (x^2 + 1)^{12} x dx = \int \frac{u^{12}}{2} du = \frac{u^{13}}{26} = \frac{(x^2 + 1)^{13}}{26} + C$$

Question 9

(a)

$$\int_0^5 (t^2 - 4) dt = \left[\frac{t^3}{3} - 4t \right]_0^5 = \frac{125}{3} - 20 = \frac{65}{3}$$

(b) Let $u = h^2$. Then $1/2 du = h dh$. The limits of integration become 4 and 9.

$$\int_2^3 e^{h^2} h dh = \int_4^9 \frac{e^u}{2} du = \left[\frac{e^u}{2} \right]_4^9 = \frac{1}{2} e^4 (e^5 - 1) \approx 4024.24$$

Question 10(a) $R = 5$ cm, $dR = 0.1$ cm

$$V = \frac{4}{3} \pi R^3$$

$$dV = 4\pi R^2 dR$$

$$dV = 4\pi(5)^2(0.1) = 10\pi$$

Therefore the volume of the ice coating is 10π .(b) Let $f(x) = |x|$. Then $f(x)$ is continuous at 0 and

$$f'(x) = \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h}$$

We are interested in $f'(0)$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

Since

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{|h|}{h} &= 1 \\ \lim_{h \rightarrow 0^-} \frac{|h|}{h} &= -1 \end{aligned}$$

Two-sided limit does not exist. Hence $f(x) = |x|$ is not differentiable at 0.