

CARLETON UNIVERSITY

FINAL
EXAMINATION
APRIL 2012

DURATION: 3 HOURS

SCANTRON SHEETS REQUIRED

Department Name and Course Number:

School of Mathematics and Statistics, MATH 3705 A, B, C

Course Instructor(s): Dr. E. Devdariani, Dr. S. Melkonian, Dr. J. Abdulrahman

AUTHORIZED MEMORANDA
Non-programmable, non-graphic calculators

This examination may be released to the Library. This examination paper may be taken from the examination room.

Instructions:

1. This examination contains 19 pages. Please report any missing pages to the proctor. Pages 17, 18 and 19 contain tables of Laplace and Fourier transforms and a summary of Fourier series.
2. This examination consists of 25 multiple-choice questions, worth 4 marks each. Enter all answers on the Scantron sheet provided.
3. Enter your name, ID number, course and section number on the Scantron sheet.
 - Section A (Dr. E. Devdariani)
 - Section B (Dr. S. Melkonian)
 - Section C (Dr. J. Abdulrahman)

1. $\mathcal{L}\{e^{2t} \cos(3t)\} =$ (a) $\frac{s-2}{(s-2)^2+9}$ (b) $\frac{s+2}{(s+2)^2+9}$ (c) $\frac{s-3}{(s-3)^2+4}$ (d) $\frac{s-2}{s^2+9}$
(e) None of these

2. $\mathcal{L}\{t \cos(2t)\} =$ (a) $\frac{4-s^2}{(s^2+4)^2}$ (b) $\frac{4s}{(s^2+4)^2}$ (c) $\frac{-4s}{(s^2+4)^2}$ (d) $\frac{s^2-4}{(s^2+4)^2}$
(e) None of these

3. $\mathcal{L}^{-1} \left\{ \frac{s+3}{s^2-6s+18} \right\} =$ (a) $e^{3t}[\cos(3t) + 6 \sin(3t)]$ (b) $e^{3t}[\cos(3t) + 2 \sin(3t)]$
(c) $e^{3t}[\cos(3t) + \sin(3t)]$ (d) $e^{-3t}[\cos(2t) + 2 \sin(3t)]$ (e) None of these

4. $\mathcal{L}^{-1} \left\{ \frac{(4s+7)e^{-2s}}{s^2+s-6} \right\} =$ (a) $u(t-2) [e^{-3t} + 3e^{2t}]$ (b) $u(t-2) [e^{3t} + 3e^{-2t}]$
(c) $u(t-2) [e^{3(t-2)} + 3e^{-2(t-2)}]$ (d) $u(t-2) [e^{-3(t-2)} + 3e^{2(t-2)}]$ (e) None of these

5. If $y(t)$ denotes the solution of the initial-value problem

$$y'' - 2y' + 5y = 2\delta(t - 2), \quad y(0) = 1, \quad y'(0) = 3,$$

then $Y(s) = \mathcal{L}\{y(t)\} =$ (a) $\frac{s+3}{(s-1)^2+4}$ (b) $\frac{s+5+2e^{-2s}}{s^2-2s+5}$ (c) $\frac{s+1+2e^{-2s}}{s^2-2s+5}$
(d) $\frac{s-1+2e^{-2s}}{(s-1)^2+4}$ (e) None of these

6. If $Y(s) = \mathcal{L}\{y(t)\} = \frac{2s+1}{s^2+s-2}$, then $y(t) =$ (a) $\frac{1}{3}(e^t - e^{-2t})$ (b) $e^{-t} + e^{2t}$
(c) $2e^{-\frac{1}{2}t} \cos\left(\frac{3}{2}t\right)$ (d) $e^t + e^{-2t}$ (e) None of these

7. The general solution of the differential equation $4x^2y'' + 8xy' + y = 0$, valid for $x \neq 0$, is given by

(a) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$ (b) $|x|^{-\frac{1}{2}}(c_1 + c_2 \ln|x|)$ (c) $c_1|x|^{-1+2\sqrt{3}} + c_2|x|^{-1-2\sqrt{3}}$
(d) $|x|^{-1}[c_1 \cos(2\sqrt{3} \ln|x|) + c_2 \sin(2\sqrt{3} \ln|x|)]$ (e) None of these

8. The general solution of the differential equation $x^2y'' + 5xy' + 13y = 0$, valid for $x \neq 0$, is given by

(a) $|x|^{-\frac{5}{2}} \left[c_1 \cos\left(\frac{3\sqrt{3}}{2} \ln|x|\right) + c_2 \sin\left(\frac{3\sqrt{3}}{2} \ln|x|\right) \right]$ (b) $e^{-2x}[c_1 \cos(3x) + c_2 \sin(3x)]$
(c) $c_1|x| + c_2|x|^{-5}$ (d) $|x|^{-2}[c_1 \cos(3 \ln|x|) + c_2 \sin(3 \ln|x|)]$ (e) None of these

9. The coefficient recursion relation of the solution $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$ of the differential equation $x^2 y'' + (x^2 - x)y' + y = 0$ is

(a) $a_{n+1} = \frac{(n+1)a_n}{n^2}$ (b) $a_{n+1} = \frac{n^2 a_n}{n+1}$ (c) $a_{n+1} = \frac{a_n}{n+1}$ (d) $a_{n+1} = \frac{-a_n}{n+1}$

(e) None of these

10. The solution of the coefficient recursion relation $a_{n+1} = \frac{2a_n}{(n+1)^2}$, $n \geq 0$, is $a_n =$

(a) $\frac{2^n a_0}{(n!)^2}$ (b) $\frac{2a_0}{(n+1)^2}$ (c) $\frac{2^n a_0}{(n+1)!}$ (d) $\frac{2^n a_0}{[(n+1)!]^2}$ (e) None of these

11. One solution y_1 of the differential equation $x^2y'' + (x^2 + 3x)y' + y = 0$ has the form

(a) $\sum_{n=0}^{\infty} a_n x^n$ (b) $\sum_{n=0}^{\infty} a_n x^{n+1}$ (c) $\sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$ (d) $\sum_{n=0}^{\infty} a_n x^{n-1}$ (e) None of these

12. The general solution of the differential equation $x^2y'' + xy' + (3x^2 - 4)y = 0$, valid for $x > 0$, is given by

(a) $c_1 J_2(\sqrt{3}x) + c_2 J_{-2}(\sqrt{3}x)$ (b) $c_1 J_{\sqrt{3}}(2x) + c_2 J_{-\sqrt{3}}(2x)$
(c) $c_1 J_2(\sqrt{3}x) + c_2 Y_2(\sqrt{3}x)$ (d) $c_1 J_{\sqrt{3}}(2x) + c_2 Y_{\sqrt{3}}(2x)$ (e) None of these

13. At $x = 59$, the Fourier sine series of $f(x) = \left\{ \begin{array}{ll} 2, & 0 \leq x < 1 \\ 4, & 1 \leq x \leq 3 \end{array} \right\}$ on $[0, 3]$ converges to
- (a) 3 (b) -3 (c) 4 (d) -4 (e) -2

14. Let $f(x) = \left\{ \begin{array}{ll} x, & 0 \leq x \leq 1 \\ -x, & -1 \leq x \leq 0 \end{array} \right\}$, and $f(x+2) = f(x)$ for all x . The Fourier series of f is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\pi x) + b_n \sin(n\pi x)],$$

where

(a) $a_0 = 1, a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1], b_n = \frac{-2(-1)^n}{n\pi}, n \geq 1$

(b) $a_0 = \frac{1}{2}, a_n = \frac{1}{n^2\pi^2}[(-1)^n - 1], b_n = \frac{(-1)^{n-1}}{n\pi}, n \geq 1$

(c) $a_n = 0, n \geq 0, b_n = \frac{-2(-1)^n}{n\pi}, n \geq 1$

(d) $a_n = 0, n \geq 0, b_n = \frac{2(-1)^n}{n\pi}, n \geq 1$

(e) $a_0 = 1, a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1], b_n = 0, n \geq 1$

15. The solution of the heat equation $u_{xx} = u_t$, $0 < x < 1$, $t > 0$, which satisfies the boundary conditions $u(0, t) = u(1, t) = 0$ and the initial condition $u(x, 0) = x$, is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where $b_n =$

- (a) $\frac{-2(-1)^n}{n\pi}$ (b) $\frac{2(-1)^n}{n\pi}$ (c) $\frac{-2(-1)^n}{n^2\pi^2}$ (d) $\frac{-2(-1)^n}{n^2\pi^2}$ (e) None of these

16. The solution of the wave equation $u_{xx} = \frac{1}{4}u_{tt}$, $0 < x < 3$, $t > 0$, which satisfies the boundary conditions $u(0, t) = 0$ and $u(3, t) = 0$, and the initial conditions $u(x, 0) = 0$ and $u_t(x, 0) = 2 \sin(\pi x) - 3 \sin(2\pi x)$, is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right) \left[a_n \cos\left(\frac{2n\pi t}{3}\right) + b_n \sin\left(\frac{2n\pi t}{3}\right) \right],$$

where

(a) $a_1 = 2$, $a_2 = -3$, $a_n = 0$ otherwise, $b_n = 0$ for all $n \geq 1$

(b) $a_3 = 2$, $a_6 = -3$, $a_n = 0$ otherwise, $b_n = 0$ for all $n \geq 1$

(c) $b_1 = 2$, $b_2 = -3$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

(d) $b_3 = 2$, $b_6 = -3$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

(e) $b_3 = \frac{1}{\pi}$, $b_6 = \frac{-3}{4\pi}$, $b_n = 0$ otherwise, $a_n = 0$ for all $n \geq 1$

17. The solution $u(x, y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ within the rectangular region $0 < x < 3$, $0 < y < 2$, subject to the boundary conditions $u(0, y) = 0$, $u(3, y) = 0$, $u(x, 0) = 3x - x^2$, $u(x, 2) = 0$, has the form

$$(a) \quad u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{3}\right) \sin\left(\frac{n\pi x}{3}\right)$$

$$(b) \quad u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(2-y)}{3}\right] \sin\left(\frac{n\pi x}{3}\right)$$

$$(c) \quad u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$$

$$(d) \quad u(x, y) = \sum_{n=1}^{\infty} a_n \sinh\left[\frac{n\pi(3-x)}{2}\right] \sin\left(\frac{n\pi y}{2}\right)$$

$$(e) \quad u(x, y) = \alpha x + \beta y + \gamma xy + \delta$$

18. The bounded solution of Laplace's equation $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ outside the circle $r = 3$, which satisfies the boundary condition $u(3, \theta) = 3 + 2 \sin(2\theta) - \cos(3\theta)$, is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

where

(a) $a_0 = 6, a_3 = -27, b_2 = 18, a_n = b_n = 0$ otherwise

(b) $a_0 = 6, a_3 = -1, b_2 = 2, a_n = b_n = 0$ otherwise

(c) $a_0 = 6, a_3 = \frac{-1}{27}, b_2 = \frac{2}{9}, a_n = b_n = 0$ otherwise

(d) $a_0 = 6, a_2 = \frac{2}{9}, b_3 = \frac{-1}{27}, a_n = b_n = 0$ otherwise

(e) $a_0 = 6, a_2 = 18, b_3 = -27, a_n = b_n = 0$ otherwise

19. The differential equation $xy'' + 2y' + xy + \lambda xy = 0$, when placed in the Sturm-Liouville form $[p(x)y']' - q(x)y + \lambda r(x)y = 0$, has the weight function $r(x) =$
- (a) 1 (b) x (c) x^2 (d) xe^{2x} (e) None of these

20. Given the Bessel identity $\frac{1}{\alpha} \frac{d}{dx} [x^\nu J_\nu(\alpha x)] = x^\nu J_{\nu-1}(\alpha x)$, $\nu > 0$, $\alpha \neq 0$,

$$\int_0^2 x^4 J_1(3x) dx = \quad (a) \frac{16}{9} [3J_2(6) - J_3(6)] \quad (b) 16J_1(6) \quad (c) \frac{32}{5} J_2(6)$$

- (d) $16[J_2(6) - J_3(6)]$ (e) None of these

21. The eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < 2, \quad y(0) = 0, \quad y'(2) = 0,$$

are

(a) $\lambda_n = \frac{n\pi}{2}, y_n = B_n \sin\left(\frac{n\pi x}{2}\right), n \geq 1$

(b) $\lambda_n = \frac{n\pi}{2}, y_n = A_n \cos\left(\frac{n\pi x}{2}\right), n \geq 0$

(c) $\lambda_n = \frac{(2n+1)^2\pi^2}{16}, y_n = B_n \sin\left[\frac{(2n+1)\pi x}{4}\right], n \geq 0$

(d) $\lambda_n = \frac{(2n+1)^2\pi^2}{16}, y_n = A_n \cos\left[\frac{(2n+1)\pi x}{4}\right], n \geq 0$

(e) $\lambda_n = \frac{(2n+1)^2\pi^2}{4}, y_n = A_n \cos\left[\frac{(2n+1)\pi x}{2}\right], n \geq 0$

22. $\mathcal{F} \{e^{-2ix-|x-3|}\} =$ (a) $\frac{2e^{-3i(\lambda-2)}}{1 + (\lambda - 2)^2}$ (b) $\frac{2e^{3i(\lambda-2)}}{1 + (\lambda - 2)^2}$ (c) $e \frac{2e^{-3i(\lambda+2)}}{1 + (\lambda + 2)^2}$
(d) $\frac{2e^{3i(\lambda+2)}}{1 + (\lambda + 2)^2}$ (e) None of these

23. $\mathcal{F} \{2xe^{-x^2}\} =$ (a) $i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$ (b) $2\lambda e^{-\lambda^2}$ (c) $\sqrt{\pi}e^{-\frac{\lambda^2}{4}}$ (d) $-i\sqrt{\pi}\lambda e^{-\frac{\lambda^2}{4}}$
(e) None of these

24. $\mathcal{F}^{-1} \left\{ \frac{e^{-3i\lambda}}{1 + (\lambda + 2)^2} \right\} =$ (a) $\frac{1}{2}e^{2i(x+3)-|x+3|}$ (b) $\frac{1}{2}e^{-2i(x+3)-|x+3|}$
(c) $\frac{1}{2}e^{-2i(x-3)-|x-3|}$ (d) $\frac{1}{2}e^{2i(x-3)-|x-3|}$ (e) None of these

25. $\mathcal{F}^{-1} \{ \lambda e^{-|\lambda|} \} =$ (a) $\frac{-i}{\pi(1+x^2)^2}$ (b) $\frac{i}{\pi(1+x^2)^2}$ (c) $\frac{-2ix}{\pi(1+x^2)^2}$
(d) $\frac{2ix}{\pi(1+x^2)^2}$ (e) None of these

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt, \quad s > 0$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p > -1, \quad \text{and } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s), \quad s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0), \quad n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n} F(s), \quad n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(x) dx$$

$$\mathcal{L}\left\{\int_0^t f(x) dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x) dx\right\} = F(s)G(s), \quad \text{where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}, \quad a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) dt \text{ whenever } f \text{ is periodic with period } T$$

Summary of Fourier Series

1. The Fourier sine series of a function f defined on $[0, L]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

2. The Fourier cosine series of a function f defined on $[0, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0.$$

3. The full Fourier series of a function f defined on $[-L, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. The full Fourier series of a function f defined on $[0, 2L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \quad b_n = \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

5. The Fourier series of a T -periodic function f on $\mathbb{R} = (-\infty, \infty)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right],$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2n\pi x}{T}\right) dx \\ &= \frac{2}{T} \int_{\alpha}^{\alpha+T} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx, \quad n \geq 0, \\ b_n &= \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2n\pi x}{T}\right) dx \\ &= \frac{2}{T} \int_{\alpha}^{\alpha+T} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx, \quad n \geq 1, \end{aligned}$$

where $L = \frac{T}{2}$ and α is any real number.

Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x} d\lambda$$

$$\mathcal{F}\{u(x-a) - u(x-b)\} = \frac{1}{i\lambda} [e^{i\lambda b} - e^{i\lambda a}], \quad a < b$$

$$\mathcal{F}\{u(x+b) - u(x-b)\} = \frac{1}{i\lambda} [e^{i\lambda b} - e^{-i\lambda b}] = \frac{2}{\lambda} \sin(\lambda b)$$

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{1 + \lambda^2}$$

$$\mathcal{F}\{e^{iax} f(x)\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a} \widehat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda \widehat{f}(\lambda)$$

$$\mathcal{F}\{xf(x)\} = -i \frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\{f(\alpha x)\} = \frac{1}{|\alpha|} \widehat{f}\left(\frac{\lambda}{\alpha}\right), \quad \alpha \neq 0$$

$$\mathcal{F}\{e^{-tx^2}\} = \sqrt{\frac{\pi}{t}} e^{-\frac{\lambda^2}{4t}}, \quad t > 0$$

$$\mathcal{F}\{(f * g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s)ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda), \quad \text{where } \widehat{g}(\lambda) = \mathcal{F}\{g(x)\}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$