Section 4.7, p. 243, #10:

$$b_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 2 & 9 & -12 & 2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 0 & \frac{15}{2} & -15 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 4 & 3 & 6 & 4 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 4 & 0 & 12 & 4 \\ 0 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$b_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix} = 3\begin{bmatrix} 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 9 \end{bmatrix} \Rightarrow [b_1]_C = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad 1$$

$$b_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 4 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 9 \end{bmatrix} \Rightarrow [b_2]_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 1$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \quad 1$$

$$\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \end{bmatrix}. \quad 2$$

Section 4.7, p. 243 #14

$B = \{ b_1 = 1 - 3t^2,\ b_2 = 2 + t - 5t^2,\ b_3 = 1 + 2t \}$

Set $C = \{1, t, t^2\}$.

$[b_1]_C = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $[b_2]_C = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, $[b_3]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix} \quad 3$$

$$\underset{B \leftarrow C}{P} = \left(\underset{C \leftarrow B}{P}\right)^{-1} = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix}$$

$$[t^2]_B = \underset{B \leftarrow C}{P} [t^2]_C = \begin{bmatrix} 10 & -5 & 3 \\ -6 & 3 & -2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}. \quad 2$$

Verify: $3(1 - 3t^2) - 2(2 + t - 5t^2) + 1 + 2t$

$= \cancel{3} - \cancel{9t^2} - \cancel{4} - \underline{2t} + \cancel{10}t^2 + \cancel{1} + \underline{2t}$

$= t^2$ ✓

Section 5.2, p. 280, Question # 18.

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

2

$$A - 4I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2

If $h = -3$, then $\dim E_4 = 2$

(If $h \neq -3$, then $\dim E_4 = 1$)

Question 4:

$$A = \begin{bmatrix} -2 & 0 & 3 & 3 \\ -9 & 1 & 9 & 9 \\ 3 & 0 & -2 & -3 \\ -9 & 0 & 9 & 10 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & 0 & 3 & 3 \\ -9 & 1-\lambda & 9 & 9 \\ 3 & 0 & -2-\lambda & -3 \\ -9 & 0 & 9 & 10-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -2-\lambda & 3 & 3 \\ 3 & -2-\lambda & -3 \\ -9 & 9 & 10-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[-(2+\lambda)(\lambda^2-8\lambda+7) - 3(-3\lambda+3) - 9(3\lambda-3)\right]$$

$$= (1-\lambda)\left[-(2+\lambda)(\lambda-1)(\lambda-7) + 9(\lambda-1) - 27(\lambda-1)\right]$$

$$= (1-\lambda)(\lambda-1)\left[-(\lambda^2-5\lambda-14) + 9 - 27\right]$$

$$= (1-\lambda)(\lambda-1)(-(\lambda^2-5\lambda+4))$$

$$= -(\lambda-1)^2(-1)(\lambda-1)(\lambda-4)$$

$$= (\lambda-1)^3(\lambda-4) = \underbrace{\lambda^4 - 7\lambda^3 + 15\lambda^2 - 13\lambda + 4.}_{3}$$

1

(a) $P(\lambda) = (\lambda-1)^3(\lambda-4) \Rightarrow \lambda = 1, 1, 1, 4.$

(b) $\lambda = 1$:

$$A - I = \begin{bmatrix} -3 & 0 & 3 & 3 \\ -9 & 0 & 9 & 9 \\ 3 & 0 & -3 & -3 \\ -9 & 0 & 9 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda I)x = 0 \Leftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

A basis for $E_1$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ 3

$\underline{\lambda = 4}$:

$$A - 4I = \begin{bmatrix} -6 & 0 & 3 & 3 \\ -9 & -3 & 9 & 9 \\ 3 & 0 & -6 & -3 \\ -9 & 0 & 9 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 0 & 1 & 1 \\ -3 & -1 & 3 & 3 \\ -3 & 0 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 0 & -3 & -1 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A-4I)x=0 \Leftrightarrow x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} \frac{1}{3}x_4 \\ x_4 \\ -\frac{1}{3}x_4 \\ x_4 \end{bmatrix}$$

A basis for $E_4$ is $\left\{\begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix}\right\}$. 1

(c) $P=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ 1, $D=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ 1

$A=PDP^{-1}$

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Verify: $AP=\begin{bmatrix} -2 & 0 & 3 & 3 \\ -9 & 1 & 9 & 9 \\ 3 & 0 & -2 & -3 \\ -9 & 0 & 9 & 10 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 12 \end{bmatrix}=PD.$$

## Section 5.4, p. 293, #8

$$[4b_1 - 3b_2]_B = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \quad (1)$$

$$\Rightarrow [T(4b_1 - 3b_2)]_B = [T]_B [4b_1 - 3b_2]_B$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix} \quad (2)$$

$$\Rightarrow T(4b_1 - 3b_2) = 0\,b_1 + 5b_2 - 5b_3$$

$$= 5b_2 - 5b_3 . \quad (1)$$

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OR $[T]_B = \left[ [T(b_1)]_B \;\; [T(b_2)]_B \;\; [T(b_3)]_B \right]$.

Thus $[T(b_1)]_B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow T(b_1) = 2b_2 + b_3,$ (2)

$[T(b_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Rightarrow T(b_2) = b_2 + 3b_3.$ (2)

$$T(4b_1 - 3b_2) = 4T(b_1) - 3T(b_2)$$

$$= 4(2b_2 + b_3) - 3(b_2 + 3b_3)$$

$$= 5b_2 - 5b_3 . \quad (1)$$