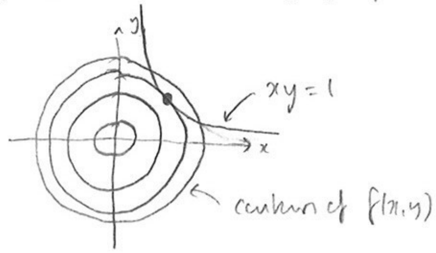
Lagrange Multipliers

- Sometimes we have to maximize / minimize a function $f(x,y)$ subject to a constraint on the allowable values of x and y, which we write as $g(x,y)=k$ (i.e. x and y lie on a particular level curve of g)

  eg. Find maximum of $f(x,y) = 1 - x^2 - y^2$ subject to $g(x,y) = xy = 1$ (only consider $x>0$ for simplicity)



Think of as 'what is the highest point while walking along the curve $xy=1$?'

At the maximum pt, the path we walk along must be parallel to the contours of f. This means ∇g must point in the same direction as ∇f at that point, so $\nabla f = \lambda \nabla g$ for some constant λ.

- To find maxima / minima of $f(x,y)$ subject to the constraint $g(x,y)=k$, set $\boxed{\nabla f = \lambda \nabla g}$, where λ is the Lagrange multiplier. Solve these (2) equations + the constraint $g(x,y)=k$ to find x, y and λ. Check the values of f to determine if the point is max / min

eg. For the example above $\nabla f = \lambda \nabla g \Rightarrow$

$$\left.\begin{array}{lll} f_x = \lambda g_x & \Rightarrow & (1)\ -2x = \lambda y \\ f_y = \lambda g_y & & (2)\ -2y = \lambda x \\ & & \&\ (3)\ xy = 1 \end{array}\right\} \text{ solve for } x, y, \lambda.$$

substitute $y = 1/x$ from (3) $\Rightarrow$ $\left.\begin{array}{l} -2x^2 = \lambda \\ -2 = \lambda x^2 \end{array}\right\} \Rightarrow -2 = -2x^4 \Rightarrow x^4 = 1 \Rightarrow x = 1$ or $x = -1$

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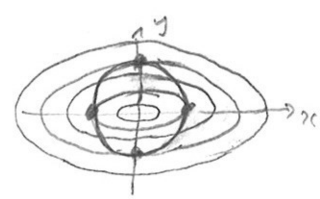
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y=1 \qquad\quad y=-1$

At $(1,1)$, $f(1,1) = -1$. Looking at the function / contour plot this is clearly a maximum

eg. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2+y^2=1$



$$\nabla f = \lambda \nabla g \Rightarrow \begin{array}{l} (1)\ 2x = \lambda 2x \\ (2)\ 4y = \lambda 2y \\ \&\ (3)\ x^2+y^2=1 \end{array} \Rightarrow \left.\begin{array}{lll} x = \lambda x & \Rightarrow & \text{need } x=0 \text{ or } \lambda = 1 \\ 2y = \lambda y & \Rightarrow & \text{need } y = 0 \text{ or } \lambda = 2 \end{array}\right\}$$

Since $x^2+y^2=1$, either $x=0, y=\pm 1, \lambda = 2$
or $y=0, x=\pm 1$ $\lambda = 1$

So extreme points are $(0,1)$ where $f(0,1) = 2 \Rightarrow$ maximum [also at $(0,-1)$, where $f=2$]

$(1,0)$ where $f(1,0)=1 \Rightarrow$ minimum [also at $(-1,0)$, where $f=1$]

eg. extension. What are the extreme values of $f(x,y) = x^2+2y^2$ on the disk $x^2+y^2 \le 1$

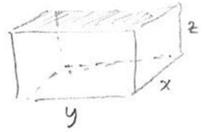
- We just found max / min on the boundary $x^2+y^2=1$ were $f=2$ and $f=1$.
- Must also check for critical pts in the interior

  $f_x = 2x = 0$ & $f_y = 4y = 0 \Rightarrow (0,0)$ is a critical pt, where $f(0,0)=0$

  $\Rightarrow$ Absolute max is 2, min is 0.

- The same method works for functions of three variables $f(x,y,z)$ subject to a constraint $g(x,y,z)=k$. In that case $\nabla f = \lambda \nabla g$ gives 3 equations + the constraint $\Rightarrow$ 4 equations to solve for x, y, z, and λ.

eg. Revisit the question what is the largest volume of a rectangular box made with $12m^2$ of cardboard.



Volume $\underbrace{V = xyz}_{f(x,y,z)}$, Area $\underbrace{2xz + 2yz + xy}_{g(x,y,z)\,=\,12} = 12$

Use Lagrange multipliers $\nabla f = \lambda \nabla g \Rightarrow$

$$\left.\begin{aligned} &(1)\quad yz = \lambda(2z+y) \\ &(2)\quad xz = \lambda(2z+x) \\ &(3)\quad xy = \lambda(2x+2y) \\ \&\ &(4)\quad 2xz + 2yz + xy = 12 \end{aligned}\right\} \text{ solve for } x, y, z \text{ and } \lambda.$$

Note $xyz = x \times (1) = y \times (2) = z \times (3)$

$= \lambda(2xz + xy) = \lambda(2yz + xy) = \lambda(2xz + 2yz)$

$\Rightarrow$ [subtract from each other] $2\lambda z(y-x) = 0$ & $\lambda y(2z - x) = 0$

$\lambda = 0$ gives $xyz = 0$ which is clearly not the maximum.

So $z(y-x) = 0$ & $y(2z-x) = 0$ $\Rightarrow$ $\underline{y = x = 2z}$ (the alternatives $z=0$ / $y=0$ give minimum)

Substitute $y = x = 2z$ into (4) $\Rightarrow$ $4z^2 + 4z^2 + 4z^2 = 12$ $\Rightarrow$ $z = 1$, $x = y = 2$. $\Rightarrow$ $\underline{V = 4}$ is max

eg. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse that are nearest and furthest from the origin.

i.e. find max / min of $d^2 = x^2 + y^2 + z^2$. [when dealing with distances, it is always easier to maximize / minimize the square d^2 instead of d].

- Since the pts lie on the plane, $\underline{z = 1 - \frac{1}{2}x - \frac{1}{2}y}$, so we must find the max / min of

$$f(x,y) = x^2 + y^2 + \left(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right)^2 \quad \text{subject to} \quad g(x,y) = x^2 + y^2 + \tfrac{1}{2}x + \tfrac{1}{2}y = 1$$

$\nabla f = \lambda \nabla g \Rightarrow$

$$\left.\begin{aligned} &(1)\ 2x - \left(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right) = \lambda\left(2x + \tfrac{1}{2}\right) \\ &(2)\ 2y - \left(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right) = \lambda\left(2y + \tfrac{1}{2}\right) \end{aligned}\right\} \text{[subtract]} \quad 2(y-x) = 2\lambda(y-x) \Rightarrow \lambda = 1 \text{ or } y = x$$

& (3) $x^2 + y^2 + \frac{1}{2}x + \frac{1}{2}y = 1$

If $\lambda = 1$ then (1) $\Rightarrow$ $\frac{1}{2}x + \frac{1}{2}y = \frac{3}{2}$, so (3) $\Rightarrow$ $x^2 + y^2 = -\frac{1}{2}$, which is impossible, #

So $y = x$, then (3) $\Rightarrow$ $2x^2 + x - 1 = 0$ $\Rightarrow$ $(2x-1)(x+1) = 0$ $\Rightarrow$ $x = -1$, $y = -1$, $z = 2$ or $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{2}$

At $(-1,-1,2)$, $f = 6$ so $d = \sqrt{6}$ $\Rightarrow$ Max

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $f = \frac{3}{4}$ so $d = \frac{\sqrt{3}}{2}$ $\Rightarrow$ Min

[Note, another way to do this question is to maximize / minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to 2 constraints, $g(x,y,z) = x + y + 2z = 2$, & $h(x,y,z) = z - x^2 - y^2 = 0$. For two constraints, add another Lagrange multiplier μ: $\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$, then solve these 3 equations + two constraints for x, y, z, λ and μ.]