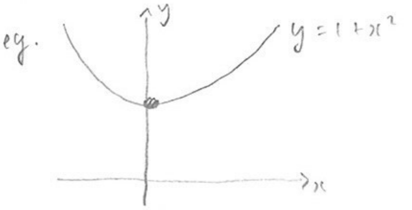
Maxima, Minima & Critical points

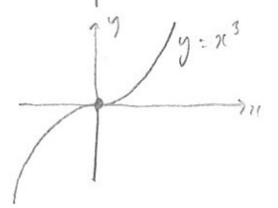
- For functions of one variable we find local max and min of $y = f(x)$ by looking for where $f'(x) = 0$ (this locates critical pts or stationary pts). Then looking at the second derivative.

eg.



$f'(x) = 2x = 0$ at $x = 0$

$f''(x) = 2 > 0 \Rightarrow x = 0$ is a minimum



$f'(x) = 3x^2 = 0$ at $x = 0$

$f''(x) = 6x = 0$ at $x = 0 \Rightarrow$ second derivative test inconclusive
in fact, from graph, $x = 0$ is neither max or min.

- For functions of two variables, $z = f(x, y)$, at a local max / min we must have $\boxed{f_x = 0 \text{ and } f_y = 0}$.
  Such points are called critical pts - as for one variable, they are not necessarily a max / min.
- The second derivative test involves f_{xx}, f_{yy} and f_{xy} at the critical pt.

Look at the quantity $D = f_{xx} f_{yy} - f_{xy}^2$. If

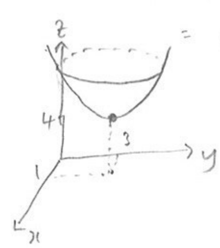
$$\begin{cases} D > 0, \ f \text{ is max/min} \begin{cases} f_{xx} > 0 \Rightarrow \text{min} \\ f_{xx} < 0 \Rightarrow \text{max} \end{cases} \\ D < 0 \Rightarrow f \text{ is a saddle point (neither max or min)} \\ D = 0 \Rightarrow \text{test is inconclusive} \end{cases}$$

eg. $z = f(x,y) = x^2 + y^2 - 2x - 6y + 14$

$$\left.\begin{aligned} f_x &= 2x - 2 = 0 \\ f_y &= 2y - 6 = 0 \end{aligned}\right\} \Rightarrow x = 1, y = 3$$

$(1,3)$ is a critical pt.

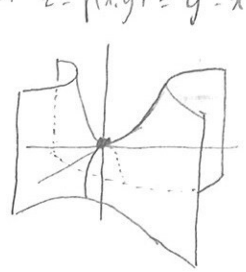
Note this is $(x-1)^2 - 1 + (y-3)^2 - 9 + 14$
$= (x-1)^2 + (y-3)^2 + 4$
paraboloid



$f_{xx} = 2, \ f_{yy} = 2, \ f_{xy} = 0$

$D = f_{xx} f_{yy} - f_{xy}^2 = 4 > 0$ so max/min. $f_{xx} > 0 \Rightarrow (1,3)$ is a minimum pt

eg. $z = f(x,y) = y^2 - x^2$



$$\left.\begin{aligned} f_x &= -2x \\ f_y &= 2y \end{aligned}\right\} \quad (0,0) \text{ is a critical pt.}$$

$f_{xx} = -2, \ f_{yy} = 2, \ f_{xy} = 0 \Rightarrow D = -4 < 0 \Rightarrow (0,0)$ is a saddle pt.

[Note, whenever f_{xx} and f_{yy} have different signs, D must be negative so the pt is a saddle. However there are some saddle pts where f_{xx} and f_{yy} have the same sign] eg. Find the local minimum and maximum values, and saddle pts, for $f(x,y) = x^4 + y^4 - 4xy + 1$

(look at p972 in book for a graph of this function)

$$\left.\begin{array}{l} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{array}\right\} \quad \left.\begin{array}{l} x^3 = y \\ y^3 = x \end{array}\right\} \quad x^9 = x \Rightarrow x(x^8 - 1) = 0 \text{ so } x = 0 \text{ or } x = 1 \text{ or } x = -1$$

$$\begin{array}{ccc} x=0 & x=1 & x=-1 \\ \downarrow & \downarrow & \downarrow \\ y=0 & y=1 & y=-1 \\ \downarrow & \downarrow & \downarrow \\ z=1 & z=-1 & z=-1 \end{array}$$

critical pts: $(0,0), (1,1), (-1,-1)$

$$\left.\begin{array}{l} f_{xx} = 12x^2 \\ f_{yy} = 12y^2 \\ f_{xy} = -4 \end{array}\right\} \quad D = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

So, for
$(0,0)$: $D = -16 < 0 \Rightarrow$ saddle pt.
$(1,1)$: $D = 128 > 0$, $f_{xx} > 0 \Rightarrow$ minimum pt.
$(-1,-1)$: $D = 128 > 0$, $f_{xx} > 0 \Rightarrow$ minimum

eg. Find the shortest distance from $(1,0,-2)$ to the plane $x + 2y + z = 4$

- 2 ways: use vectors or use calculus.

[recall, using vectors, choose a pt in the plane (eg. $(4,0,0)$) to find a vector from the point to the plane: $\langle 4,0,0\rangle - \langle 1,0,-2\rangle = \langle 3,0,2\rangle$. Take component in normal direction $\hat{n} = \langle 1,2,1\rangle$

$$\Rightarrow \quad \frac{\langle 3,0,2\rangle \cdot \langle 1,2,1\rangle}{\sqrt{6}} = \frac{5}{\sqrt{6}}$$]

Using calculus, try to minimize the square of the distance from $(1,0,-2)$ to a point $(x, y, 4-x-2y)$ in the plane:

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2 \qquad [d^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]$$

$$\begin{array}{lll} f_x = 2(x-1) - 2(6-x-2y) = 0 & 4x + 4y = 14 & 2x + 2y = 7 \\ f_y = 2y - 4(6-x-2y) = 0 & 4x + 10y = 24 & 2x + 5y = 12 \end{array} \Bigg\} \quad 3y = 5 \Rightarrow y = \frac{5}{3}$$

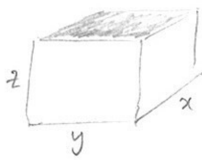
$$x = \frac{7}{2} - y = \frac{21-10}{6} = \frac{11}{6}$$

So $\left(\frac{11}{6}, \frac{5}{3}\right)$ is the critical pt for f. Intuitively, there must be a minimum distance, so since there is only one critical pt, it must be the minimum. If we were unsure, we could check:

[$f_{xx} = 4$, $f_{yy} = 10$, $f_{xy} = 4$, so $D = f_{xx}f_{yy} - f_{xy}^2 = 24 > 0$, $f_{xx} > 0 \Rightarrow$ minimum]

At $x = \frac{11}{6}$, $y = \frac{5}{3}$, the distance is $d^2 = \left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(6 - \frac{11}{6} - \frac{10}{3}\right)^2 = \left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2 = \frac{5^2}{6}$, so $d = \frac{5}{\sqrt{6}}$

eg. A rectangular box with no lid is to be made with $12\,m^2$ of cardboard. What is the maximum possible volume for the box?



- Write the side lengths as x, y, z

The volume is $V = xyz$

The area of the sides are xz for two of them, yz for the other two, and the area of the bottom is xy

So total area is: $2xz + 2yz + xy = 12 \leftarrow$ Use this to write $z = z(x,y) = \dfrac{12 - xy}{2(x+y)}$

The volume is then $V(x,y) = \dfrac{xy(12-xy)}{2(x+y)}$ and we have to maximize this function.

$$V_x = \frac{y(12-xy) - xy^2}{2(x+y)} - \frac{xy(12-xy)}{2(x+y)^2} = \frac{y^2(12-xy) - xy^2(x+y)}{2(x+y)^2} = \frac{y^2(12-2xy-x^2)}{2(x+y)^2} = 0$$

$$V_y = \frac{x(12-xy) - x^2y}{2(x+y)} - \frac{xy(12-xy)}{2(x+y)^2} = \text{(as above with } x/y \text{ swapped)} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2} = 0$$

Clearly $x=0$ or $y=0$ will not give the max volume, so we need
$$\left.\begin{array}{l} 12 - 2xy - x^2 = 0 \\ 12 - 2xy - y^2 = 0 \end{array}\right\} \; x^2 = y^2 \Rightarrow x = y$$
(x, y must be positive)

$12 - 3x^2 = 0$

$\Rightarrow x = 2, y = 2 \Rightarrow z = \dfrac{12-4}{2(4)} = 1.$

So at max, $x = y = 2$ and $z = 1 \Rightarrow \underline{V = 4\ m^3}$

We could check second derivatives to show this is a maximum (this would be messy algebra!) or we argue that there must intuitively be a maximum and since this is the only viable critical pt ($x=y=0$ gives the minimum $V=0$) it must be the maximum.

eg. Find and classify the critical points of $f(x,y) = x^3 + y^3 - 3xy$

$$\begin{array}{l} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0 \end{array} \Rightarrow \begin{array}{l} y = x^2 \\ x = y^2 \end{array} \Rightarrow x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow \begin{array}{l} x = 0 \\ y = 0 \end{array} \text{ or } \begin{array}{l} x = 1 \\ y = 1 \end{array}$$

critical pts $(0,0)$ & $(1,1)$

$f_{xx} = 6x$

$f_{yy} = 6y \Rightarrow D = 36xy - 9$

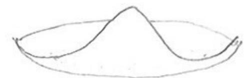
$f_{xy} = -3$

At $(0,0)$, $D = -9 < 0 \Rightarrow$ saddle pt.

At $(1,1)$, $D = 27 > 0$ & $f_{xx} = 6 > 0 \Rightarrow$ minimum

Absolute maxima & minima

- Often we have to maximize a function over a particular set of input values.
  Finding the critical pts tells where local maxima / minima are, but the overall maximum may occur at the boundary. eg. a Mexican hat shape / bottom of a wine bottle



Max may be at edge or in middle, depending on height of bump.

- To find absolute max / min values, find critical pts, then find max / min values on boundaries, then compare the values to determine which is largest / smallest.

eg. Find the absolute maximum of $y = f(x,y) = x^2 - 2xy + 2y$ over the rectangle $\{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$

- critical pts:

$$\left.\begin{aligned} f_x &= 2x - 2y = 0 \\ f_y &= -2x + 2 = 0 \end{aligned}\right\} \Rightarrow x = 1,\ y = 1 \quad \text{so } (1,1) \text{ is a critical pt.} \quad f(1,1) = 1$$

[$f_{xx} = 2$, $f_{yy} = 0$, $f_{xy} = -2$ so $D < 0 \Rightarrow$ saddle]

- look at boundaries:

$x = 0$ $\quad f(0,y) = 2y$ $\quad$ max at $y = 2$ $\quad f(0,2) = 4$

$x = 3$ $\quad f(3,y) = 9 - 4y$ $\quad$ max at $y = 0$ $\quad f(3,0) = 9$

$y = 0$ $\quad f(x,0) = x^2$ $\quad$ max at $x = 3$ $\quad f(3,0) = 9$

$y = 2$ $\quad f(x,2) = x^2 - 4x + 4 = (x-2)^2$ $\quad$ max at $x = 0$ $\quad f(0,2) = 4$

[in more complicated cases, might have to use 1 variable calculus to find maximum / minimum on the edges].

- comparing values the overall maximum is at $(3,0)$ where $f = 9$.

[Note, the critical pt at (1,1) is actually a saddle pt, not a local maximum; but since we are going to compare the value with the boundary values there is no need to go to the effort of determining this] eg. A harder example. Find & classify the critical pts of $f(x,y) = -(x^2-1)^2 - (x^2y-x-1)^2$ (old final question)

$f_x = -2(x^2-1)(2x) - 2(x^2y-x-1)(2xy-1) = 0$

$f_y = -2x^2(x^2y-x-1) = 0$ $\Rightarrow$ $x=0$ or $x^2y-x-1=0$

$x=0 \;\downarrow\; f_x = -2 \neq 0$

$x^2y-x-1=0 \;\downarrow\; f_x = -4x(x^2-1) \Rightarrow x=1$ or $x=-1$; $y=2$, $y=0$

$\Rightarrow$ critical pts $(-1,0)$ & $(1,2)$

$f_{xx} = -4(x^2-1) - 4x(2x) - 4y(x^2y-x-1) - 2(2xy-1)^2$

$f_{yy} = -4x^4$

$f_{xy} = -8x^3y + 6x^2 + 4x$

At $(-1,0)$, $f_{xx} = -8-2 = -10$

$f_{yy} = -4$

$f_{xy} = 6-4 = 2$

$D = f_{xx}f_{yy} - f_{xy}^2 = 40-4 = 36 > 0$, $f_{xx} < 0 \Rightarrow (-1,0)$ is a maximum

At $(1,2)$, $f_{xx} = -8-18 = -26$

$f_{yy} = -4$

$f_{xy} = -16+6+4 = -6$

$D = 104-36 > 0$, $f_{xx} < 0 \Rightarrow$ $(1,2)$ is also a maximum

Note that in this example there are two maxima and no minima. This shows an important difference from functions of one variable, for which there must be a local minimum in between two local maxima.