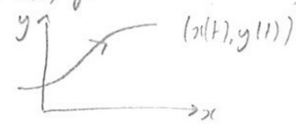
Chain Rule

- In single variable calculus, the chain rule is used for composite functions like $f(t) = \sin t^2$, which is $f(x) = \sin x$, $x(t) = t^2$.

  Then $\dfrac{df}{dt} = \dfrac{df}{dx}\dfrac{dx}{dt} = (\cos x)\,2t = 2t\cos t^2$

- In multivariable calculus, if $z = f(x,y)$ and $x(t), y(t)$ are functions of t, then $\boxed{\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}}$

  If $z = f(x,y)$ is the height at coordinates (x,y), $(x(t), y(t))$ is a path, the rate of change of height along the path is $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$.



eg. If $z = x^2 + xy - y^2$ and $x(t) = \cos t$, $y(t) = \sin 2t$, find $\frac{dz}{dt}$ when $t=0$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x+y)(-\sin t) + (x-2y)(2\cos 2t)$$

When $t=0$, $x=1$, $y=0$.

$$= (2)(0) + (1)(2)$$
$$= 2$$

- If x and y are functions of 2 variables, $x(s,t)$, $y(s,t)$, the same rule applies for the partial derivatives

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \qquad \boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

eg. $z = e^x \sin y$, $x = 2st$, $y = s^2 - t^2$. Find $\frac{\partial z}{\partial s}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(2t) + (e^x \cos y)(2s) = e^{2st}\left[2t\sin(s^2-t^2) + 2s\cos(s^2-t^2)\right].$$

- In general, when differentiating with respect to an independent variable, we must account for all possible ways the function can depend on that variable.

  [In example above s and t are the independent variables; x and y are <u>intermediate</u> variables].

eg. Temperature is a function of time and space $T = f(x,y,z,t)$. If I travel along the path $(x(t), y(t), z(t))$ what is the rate of change of temperature I feel?

The independent variable is t; x, y, z are intermediate variables. The chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t}$$

eg. A more complicated example. If $z = z(x,y)$, $x = x(s,t)$, $y = y(t)$ and $s = s(t)$, find $\frac{dz}{dt}$.

The independent variable is t; x, y and s are intermediate variables

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\frac{ds}{dt} + \frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

eg. Find the rate of change of $f(x,y)$ along a line through (x_0, y_0) in direction of unit vector $\langle a, b\rangle$.

The line is $\begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \end{cases}$ so $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = \dfrac{\partial f}{\partial x}a + \dfrac{\partial f}{\partial y}b$

Directional derivative and Gradient

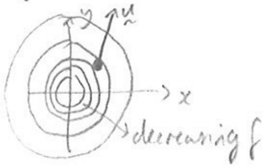
- The rate of change of $f(x,y)$ in the direction of a unit vector $\underset{\sim}{u} = \langle a,b \rangle$ is

$$\boxed{D_u f = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b \quad \text{or} \quad \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \underset{\sim}{u}}$$

This is called the directional derivative

- The vector $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$ turns out to be very important. It is called the gradient of $f(x,y)$, and is often written as $\boxed{\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle}$. If f depends on more variables, the gradient has more components; eg. for $f(x,y,z)$, $\nabla f = \langle f_x, f_y, f_z \rangle$.

eg. find the rate of change of $f(x,y) = e^{-x^2-y^2}$ in the direction $\langle 1,2 \rangle$ at the point $(1,1)$.



Need unit vector, so $\underset{\sim}{u} = \frac{1}{\sqrt{5}} \langle 1,2 \rangle$.

Then $D_u = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \frac{1}{\sqrt{5}} \langle 1,2 \rangle$

$= \langle -2x e^{-x^2-y^2}, -2y e^{-x^2-y^2} \rangle \cdot \frac{1}{\sqrt{5}} \langle 1,2 \rangle$

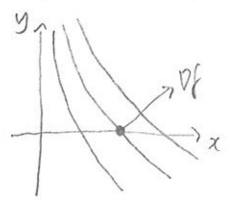
$= \langle -2e^{-2}, -2e^{-2} \rangle \cdot \frac{1}{\sqrt{5}} \langle 1,2 \rangle \quad = \quad -\frac{6}{\sqrt{5}} e^{-2}$

- The directional derivative can be written $D_u f = \nabla f \cdot \underset{\sim}{u} = |\nabla f| |\underset{\sim}{u}| \cos\theta = |\nabla f| \cos\theta$, where θ is the angle between ∇f and $\underset{\sim}{u}$. This is largest when $\theta = 0$, when it has size $|\nabla f|$.

So $\boxed{\text{The direction of } \nabla f \text{ is the direction of steepest increase of } f, \text{ and } |\nabla f| \text{ is this steepest slope}}$

- On a contour plot of $f(x,y)$, ∇f points perpendicular to the contours

eg. If $f(x,y) = xe^y$, what is the direction and size of the maximum rate of change of f from the point $(2,0)$



$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle = \langle 1, 2 \rangle$ at $(2,0)$

So maximum slope in direction $\langle 1,2 \rangle$. The size of the slope is $|\nabla f| = \sqrt{5}$

- What is the rate of change in the direction towards $(0,2)$

The direction is $\langle -2,2 \rangle$, so unit vector is $\underset{\sim}{u} = \frac{1}{\sqrt{8}} \langle -2,2 \rangle = \frac{1}{\sqrt{2}} \langle -1,1 \rangle$. So $D_u = \nabla f \cdot \underset{\sim}{u} = \frac{1}{\sqrt{2}}$

Level Curves & the Gradient as a Normal

- Any curve in $\mathbb{R}^2$ can be written as a level curve of a function $f(x,y)$ (ie a contour line). A vector pointing normal to the curve is ∇f.
- In $\mathbb{R}^3$, a surface can be considered as the level surface of a function $F(x,y,z)$. $\boxed{\text{A vector pointing normal to the surface is } \nabla F}$

eg. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ is the level surface $F(x,y,z) = 16$, for $F(x,y,z) = 4x^2 + 2y^2 + z^2$.

The normal to the surface is $\nabla F = \langle F_x, F_y, F_z \rangle = \langle 8x, 4y, 2z \rangle$

eg. Find the equation of the tangent plane to this ellipsoid at $(1,2,2)$.

A normal at that point is $\langle 8,8,4 \rangle$ or $\langle 2,2,1 \rangle$. The tangent plane is $\underset{\sim}{n} \cdot \langle x,y,z \rangle = \underset{\sim}{n} \cdot \langle 1,2,2 \rangle$ ie the known point on the plane

ie. $2x + 2y + z = 8$ [compare with same result found before using implicit differentiation for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$]