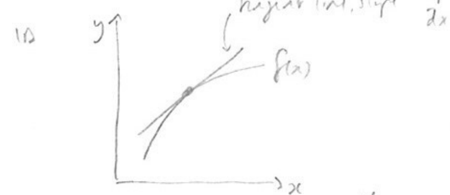
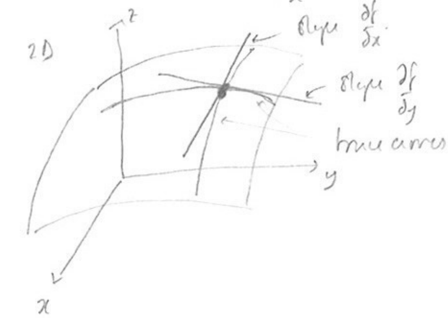
Partial Derivatives

- Recall, for $y=f(x)$ in 1D, $\frac{df}{dx}$ is the rate of change of f with respect to x, and represents the slope of the tangent to the curve at the point x.



- The partial derivative of $f(x,y)$, $\frac{\partial f}{\partial x}$, or f_x, is the rate of change of f in the x direction, and represents the slope of the tangent to the trace curve with constant y, at the point (x,y)



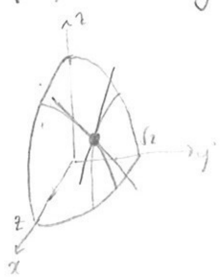
> To find $\frac{\partial f}{\partial x}$, regard y as a constant in the formula, and differentiate $f(x,y)$ with respect to x. For $\frac{\partial f}{\partial y}$, regard x as constant, and differentiate with respect to y.

eg. If $f(x,y) = x^3+2xy^3-y^2$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(2,1)$

Keep y constant $\Rightarrow \frac{\partial f}{\partial x} = 3x^2+2y^3 \Rightarrow \frac{\partial f}{\partial x}(2,1) = 12+2 = 14$. Keep x constant $\Rightarrow \frac{\partial f}{\partial y} = 6xy^2-2y \Rightarrow \frac{\partial f}{\partial y}(2,1) = 12-2 = 10$.

- The definition of the derivatives is $\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ $\quad \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$

eg. $f(x,y) = 4-x^2-2y^2$. (paraboloid). Find the slopes of the tangent lines of the trace curves through the point $(1,1,1)$



These are $\frac{\partial f}{\partial x}(1,1)$ (tangent to y-trace) and $\frac{\partial f}{\partial y}(1,1)$ (tangent to x-trace).

$\frac{\partial f}{\partial x} = -2x = -2$ at $(x,y) = (1,1)$

$\frac{\partial f}{\partial y} = -4y = -4$ at $(x,y) = (1,1)$.

eg. $z = f(x,y)$ defined implicitly by $x^3+y^3+z^3+6xyz = 1$. Calculate $\frac{\partial z}{\partial x}$.

treat y as constant, do $\frac{\partial}{\partial x}$: $3x^2+3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x}(3z^2+6xy) = -(3x^2+6yz)$ $\quad \frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy}$

eg If $f(x,y,z) = xe^{-(x^2+y^2+z^2)} - z$, what are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial z}$

for $\frac{\partial f}{\partial x}$ treat y and z as constants $\quad \frac{\partial f}{\partial x} = e^{-(x^2+y^2+z^2)} + xe^{-(x^2+y^2+z^2)}(-2x) = (1-2x^2)e^{-(x^2+y^2+z^2)}$ (use product rule/chain rule)

for $\frac{\partial f}{\partial z}$ treat x and y as constants $\quad \frac{\partial f}{\partial z} = xe^{-(x^2+y^2+z^2)}(-2z) - 1 = -2xze^{-(x^2+y^2+z^2)} - 1$

- Higher derivatives eg. $f(x,y) = x^3+2xy^3-y^2$

$f_x = 3x^2+2y^3 \qquad f_y = 6xy^2-2y$

2nd derivatives $\quad f_{xx} = 6x,\ f_{xy} = 6y^2 \qquad f_{yx} = 6y^2,\ f_{yy} = 12xy-2$

3rd derivatives $\quad f_{xxx} = 6,\ f_{xxy} = 0,\ f_{xyx} = 0,\ f_{xyy} = 12y$ etc.

etc.

Note $f_{xy} = f_{yx}$, or $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$

This is a general result, known as Clairaut's Theorem. It is true whenever f_{xy} and f_{yx} are both continuous.

- Partial differential equations

  - Laplace's equation (eg. electric / magnetic potential, steady temperature, fluid dynamics...)

    $\boxed{u_{xx} + u_{yy} = 0}$ (often written $\nabla^2 u = 0$) for $u(x,y)$

    Show that $u = e^y \sin x$ is a solution: $u_x = e^y \cos x$ $u_{xx} = -e^y \sin x$ $\Rightarrow u_{xx} + u_{yy} = 0$ ✓
    $u_y = e^y \sin x$ $u_{yy} = e^y \sin x$

  - Wave equation (eg. water waves, sound waves, light waves, guitar strings...)

    $\boxed{u_{tt} = c^2 u_{xx}}$ c is the wave speed (constant)

    Show that $u = \sin k(x-ct)$ is a solution (k, constant): $u_x = k \cos k(x-ct)$ $u_{xx} = -k^2 \sin k(x-ct)$
    $u_t = -kc \cos k(x-ct)$ $u_{tt} = -k^2 c^2 \sin k(x-ct)$
    $\Rightarrow u_{tt} = c^2 u_{xx}$ ✓

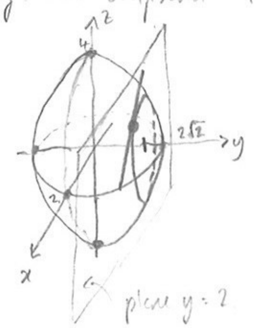
  - Heat equation (eg. heat conduction, diffusion of gas)

    $\boxed{u_t = \kappa u_{xx}}$ κ is the diffusivity (constant)

    Show that $u = e^{-\kappa t} \cos x$ is a solution: $u_t = -\kappa e^{-\kappa t} \cos x$ $\Rightarrow u_t = \kappa u_{xx}$ ✓
    $u_{xx} = -e^{-\kappa t} \cos x$

- Tangent lines and planes

eg. The ellipsoid $4x^2 + 2y^2 + z^2 = 16$ intersects $y = 2$ in an ellipse. Find the equation of the tangent to this ellipse at $(1,2,2)$



Slope of tangent on trace $y=2$ $\frac{\partial z}{\partial x}(1,2)$

to calculate $\frac{\partial z}{\partial x}$, implicit differentiation $\Rightarrow$ $8x + 0 + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{8x}{2z} = -\frac{8}{4} = -2$

Equation of tangent line $[z - z_0 = m(x - x_0)]$ is $(z-2) = -2(x-1)$, $y = 2$ or $z = 4 - 2x$, $y = 2$.

Or, in parametric form, direction of line is $\langle 1, 0, \frac{\partial z}{\partial x} \rangle = \langle 1, 0, -2 \rangle$, and it goes through $(1,2,2)$

so the line is $\underline{r} = \langle 1,2,2 \rangle + t\langle 1,0,-2 \rangle$ or $x = 1+t$, $y = 2$, $z = 2 - 2t$ $\rightarrow z = 2 - 2(x-1) = 4 - 2x$

Note the tangent line on the trace $x=1$ has slope $\frac{\partial z}{\partial y}(1,2)$ $4y + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{4y}{2z} = -\frac{8}{4} = -2$

- The tangent plane to the surface $z = f(x,y)$ at point (x_0, y_0) is the plane containing tangent lines to both x and y trace curves.

It has the equation $\boxed{z - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\,(y - y_0)}$

[Note the normal to the plane is therefore $\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$ from earlier work on planes]

eg. The tangent plane to the ellipsoid at $(1,2,2)$ is

$z - 2 = -2(x-1) - 2(y-2)$, or $z + 2x + 2y = 8$