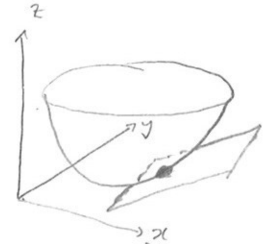
Tangent planes and linear approximations



- The equation for the tangent plane to $f(x,y)$ at (a,b) can be written as

$$z = \boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

- This can be used as an approximation to the function near (a,b) ie $f(x,y) \approx L(x,y)$ for (x,y) near (a,b).
  $L(x,y)$ is called the tangent plane approximation, or linearization of f.

- If $x = a + \Delta x$ and $y = b + \Delta y$ then we define the change in z, or increment of z, as $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$.
  The tangent plane approximation suggests $\Delta z \approx \cancel{f(a,b)} + f_x(a,b)\Delta x + f_y(a,b)\Delta y - \cancel{f(a,b)}$
  We say f is differentiable at (a,b) if this is a good approximation when Δx and Δy are small.
  Specifically, f is differentiable at (a,b) if $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ and the 'errors' ε_1 and $\varepsilon_2 \to 0$ as Δx and $\Delta y \to 0$.
- If f and its partial derivatives exist and are continuous at (a,b), then f is differentiable there.

eg. $f(x,y) = xe^{xy}$. Use the linearization of f at $(1,0)$ to approximate $f(1.1, -0.1)$

$f_x = e^{xy} + xye^{xy} = (1+xy)e^{xy}$ $\quad f_x(1,0) = 1$

$f_y = x^2 e^{xy}$ $\quad f_y(1,0) = 1$

[Note f, f_x and f_y are continuous on $\mathbb{R}^2$ so f is differentiable everywhere on $\mathbb{R}^2$].

The linear approximation is
$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y) \\ &= 1 + 1(x-1) + 1(y) \\ &= x+y \end{aligned}$$
so $f(1.1,-0.1) \approx L(1.1,-0.1) = 1$ $\quad$ [Actually $f(1.1,-0.1) = 1.1e^{-0.11} \approx 0.985\ldots$]

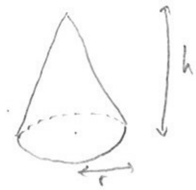
- Differentials. If we move from (x,y) to $(x+dx, y+dy)$, the tangent plane approximation suggests that z changes from $f(x,y)$ to $f(x,y) + f_x(x,y)dx + f_y(x,y)dy$ so the approximate change in z is $\boxed{dz = f_x(x,y)dx + f_y(x,y)dy}$
  This is called the differential or total differential. It can be used to approximate the change in a function due to a small change in the inputs.
  [Note dz is not the actual change $\Delta z = f(x+dx, y+dy) - f(x,y)$ but is a good approximation when dx and $dy \to 0$]

eg. If $z = f(x,y) = x^2 + 3xy - y^2$, calculate the change in z from $(x,y) = (2,3)$ to $(2.05, 2.96)$.

The differential is
$$\begin{aligned} dz &= (2x+3y)dx + (3x-2y)dy \\ &= 13dx + 0dy \quad \text{at } (x,y) = (2,3) \end{aligned}$$
If $(dx,dy) = (0.05, -0.04)$ then $dz \approx 13(0.05) = 0.65$

The actual change is
$$\begin{aligned} \Delta z &= f(2.05, 2.96) - f(2,3) \\ &= 0.6449 \end{aligned}$$
so dz is a good approximation to Δz

eg. A circular cone has base radius 10cm, height 25cm, but the errors in these measurements are 0.1cm.
Use differentials to calculate the error when calculating the volume of the cone.



- Idea: Construct formula for cone volume V as function of radius r and height h.
  Then calculate differential dV from changes in radius and height, dr and dh.
  Substitute in $r = 10cm$, $h = 25cm$ $dr = 0.1cm$ and $dh = 0.1cm$.

Volume of cone $V = \frac{\pi}{3} r^2 h = V(r,h)$

Differential $dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$

$= \frac{2\pi}{3} rh\, dr + \frac{\pi}{3} r^2 dh = \frac{2\pi}{3} 250\, dr + \frac{\pi}{3} 100\, dh$ when $r = 10$, $h = 25$

$= \frac{2\pi}{3} 25 + \frac{\pi}{3} 10$ when $dr = dh = 0.1$

$= 20\pi \approx \underline{63\ cm^3}$ is the approximate error.