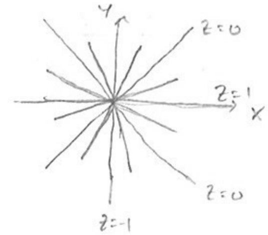
Limits & Continuity

eg. Consider $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$. What are the level curves $z=k$? $k = \dfrac{x^2-y^2}{x^2+y^2} \Rightarrow y^2(1+k) = x^2(1-k)$

$D = \mathbb{R}^2 \setminus \{(0,0)\}$ these are straight lines through the origin



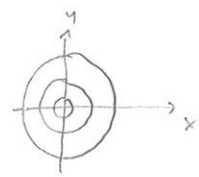
Something odd is going on at the origin:

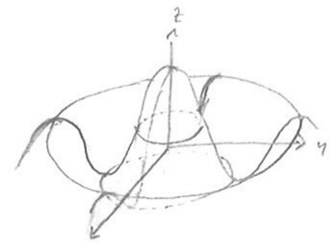
- on the x axis, $y=0$: $f(x,0) = \dfrac{x^2}{x^2} = 1$
- on the y axis, $x=0$: $f(0,y) = \dfrac{-y^2}{y^2} = -1$

These are not consistent at the origin $(x,y) = (0,0)$. We say that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

eg. Consider $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$= \dfrac{\sin(r^2)}{r^2}$ where $r^2 = x^2+y^2$

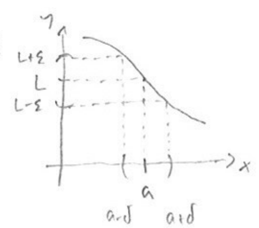




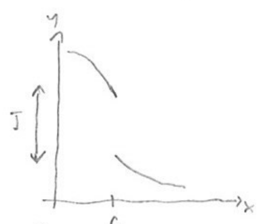
When r is small, $\sin(r^2) \approx r^2$, so $\dfrac{\sin(r^2)}{r^2} \to 1$ as $r \to 0$. It does not matter which route is taken as $(x,y)\to(0,0)$. The limit is always the same (though we won't prove that here). We write $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$

- In general, we write $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, or $f(x,y) \to L$ as $(x,y)\to(a,b)$ if the values of f approach L as $(x,y)\to(a,b)$ along any path in the domain of f.
- To show that the limit does not exist, show that $f(x,y)$ approaches different values along two different paths.
- The strict definition is that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if, for any $\varepsilon > 0$, we can choose $\delta > 0$ so that $|f(x,y) - L| < \varepsilon$ whenever $|\langle x,y\rangle - \langle a,b\rangle| < \delta$ and $(x,y) \in D$.

In 1d

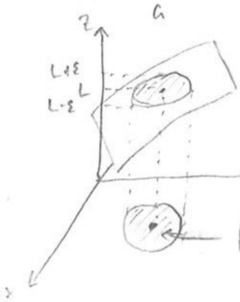


In this case, no matter how small an ε we're given we can choose δ sufficiently small.



In this case, no matter how small a region around a we choose, the values of f span a range larger than J, so if we're given $\varepsilon < J$, there is no choice of δ that works.

In 2d



$|\langle x,y\rangle - \langle a,b\rangle| < \delta$

eg. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$. Does $\lim_{(x,y)\to(0,0)}$ exist?

Consider (x,y) approaching $(0,0)$ along different paths,

on the straight lines $y = mx$, $f(x,mx) = \dfrac{m^2x^3}{x^2+m^4x^4} = \dfrac{m^2x}{1+m^4x^2} \to 0$ as $x \to 0$, regardless of m

so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ on any straight line.

But what about the line $x = y^2$? $f(y^2,y) = \dfrac{y^4}{y^4+y^4} = \dfrac{1}{2} \to \dfrac{1}{2}$ as $y \to 0$. This is not the same, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

- Note it is not possible to show that the limit does exist this way - you could never try all the possible paths. To show that a limit exists, use the ε-δ definition

eg. $f(x,y) = \dfrac{2x^2y}{x^2+y^2}$ Find $\lim_{(x,y)\to(0,0)}$ if it exists.

On the line $y = mx$, $f(x,mx) = \dfrac{2mx^3}{x^2} \to 0$ as $x \to 0$, so if it exists the limit must be 0.

Now, suppose $\varepsilon > 0$ is given. We want to find $\delta > 0$ so that if $\sqrt{x^2+y^2} < \delta$ then $|f - 0| < \varepsilon$.

If $x^2+y^2 < \delta^2$ then $y^2 < \delta^2$ so $|y| < \delta$.

Hence $\left| \dfrac{2x^2y}{x^2+y^2} \right| < 2\delta$ $\left[\text{note } \dfrac{x^2}{x^2+y^2} < 1 \right]$.

So let's choose $\delta = \varepsilon/2$, then if $\sqrt{x^2+y^2} < \delta$, $|f(x,y)| < \varepsilon$. So the limit exists and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

- A function $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say f is continuous on its domain D if it is continuous at all pts (a,b) in D.

eg. $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$, since $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

- How to show a function is continuous? Use some well-known properties
  - any polynomial is continuous
  - any sum, product, quotient of continuous functions is continuous on its domain eg. $\dfrac{xy+1}{x^2+y^2}$ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

    (note the domain doesn't usually include pts where the quotient is zero)

    [eg. a rational function is a ratio of two polynomials; these are all continuous]
  - composition of continuous functions give a continuous function;

    eg. $g(\theta) = \sin\theta$ and $h(x,y,z) = xyz$ are continuous, so $f(x,y,z) = g(h(x,y,z)) = \sin(xyz)$ is continuous on $\mathbb{R}^3$