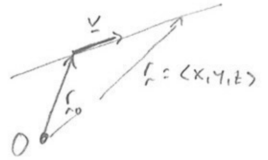
Equations of lines and planes



- A line is defined by a point on the line with position vector $\underset{\sim}{r}_0$ and a direction vector $\underset{\sim}{v}$:

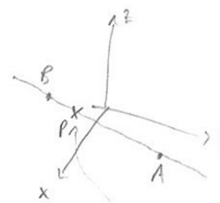
$$\boxed{\underset{\sim}{r} = \underset{\sim}{r}_0 + t\underset{\sim}{v}}$$

t is a parameter that represents distance along the line from $\underset{\sim}{r}_0$ [in units of $|\underset{\sim}{v}|$; often $\underset{\sim}{v}$ is chosen to be a unit vector, but not necessarily].

In components,
$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$
where $\underset{\sim}{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\underset{\sim}{v} = \langle a, b, c \rangle$. ← This is a <u>parametric equation</u>

- Sometimes it is helpful to eliminate t: $t = \boxed{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}$ ← These are the <u>symmetric equations</u> of a line

eg. find the line passing through $A(2,4,-3)$ and $B(3,-1,1)$



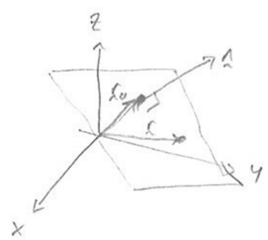
direction $\underset{\sim}{v} = \vec{AB} = \langle 1, -5, 4 \rangle$.
pt. on line $\underset{\sim}{r}_0 = \langle 2, 4, -3 \rangle$ $\Rightarrow$ $\underset{\sim}{r} = \langle 2, 4, -3 \rangle + t\langle 1, -5, 4 \rangle$

or $\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}$

eg. At what point does this line intersect the xz plane?
This occurs when $y = 4 - 5t = 0 \Rightarrow t = 4/5$, so $x = 2 + 4/5 = \frac{14}{5}$, $z = -3 + \frac{16}{5} = \frac{1}{5} \Rightarrow P\left(\frac{14}{5}, 0, \frac{1}{5}\right)$

- If a restricted range of t is allowed, this defines a line segment.
  eg the line between points with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\underset{\sim}{r} = \underset{\sim}{a} + t(\underset{\sim}{b} - \underset{\sim}{a})$ with $0 \le t \le 1$.
- Two lines are <u>skew</u> if they are not parallel and don't intersect.

- A <u>plane</u> is described by one point on the plane with position vector $\underset{\sim}{r}_0$, and by the normal vector $\underset{\sim}{n}$. The vector from $\underset{\sim}{r}_0$ to any other point on the plane, with position vector $\underset{\sim}{r}$, must be orthogonal to $\underset{\sim}{n}$, so



$$\boxed{(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{n} = 0} \quad \text{or} \quad \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{r}_0 \cdot \underset{\sim}{n}$$

In components, if $\underset{\sim}{n} = \langle a, b, c \rangle$, $\boxed{ax + by + cz = ax_0 + by_0 + cz_0}$

or $ax + by + cz + d = 0$ where $d = -(ax_0 - by_0 - cz_0) = -\underset{\sim}{r}_0 \cdot \underset{\sim}{n}$

- Any equation that is linear in x, y and z describes a plane with normal $\langle a, b, c \rangle$

eg. find the equation of the plane containing points $P(1,3,2)$ $Q(3,-1,6)$ $R(1,0,0)$

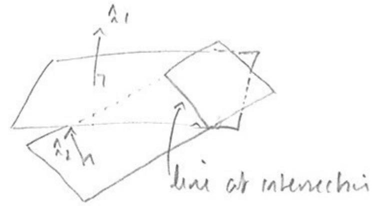
$\vec{PQ} = \langle 2, -4, 4 \rangle$ and $\vec{PR} = \langle 0, -3, -2 \rangle$ are two vectors in the plane, so a normal vector is

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 0 & -3 & -2 \end{vmatrix} = (20)\underset{\sim}{i} - (-4)\underset{\sim}{j} + (-6)\underset{\sim}{k} = \langle 20, 4, -6 \rangle$. We only care about the direction so take $\underset{\sim}{n} = \langle 10, 2, -3 \rangle$. A point on the plane is R, with position vector $\underset{\sim}{r}_0 = \langle 1, 0, 0 \rangle$, so

[We'd get the same equation by choosing P or Q for $\underset{\sim}{r}_0$] $\quad \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{r}_0 \cdot \underset{\sim}{n} \Rightarrow \underline{10(x-1) + 2y - 3z = 0}$

• The angle between two planes is the angle between their normal vectors.

If the angle is 0, the planes are parallel and don't intersect.
Otherwise the planes intersect along a line, which must be orthogonal to both normal vectors. So has direction $\underline{n}_1 \times \underline{n}_2$



eg. find the line of intersection between $x+y+z=1$ and $x-2y+3z=1$

normals are $\underline{n}_1 = \langle 1,1,1\rangle$ and $\underline{n}_2 = \langle 1,-2,3\rangle$. The line has direction $\underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\underline{i} - 2\underline{j} - 3\underline{k}$
$= \langle 5,-2,-3\rangle$

Also need a point which is on the intersection; look for the easiest one, eg. if $z=0$ then $\begin{matrix} x+y=1 \\ x-2y=1 \end{matrix} \Rightarrow y=0, x=1$

so $\underline{r}_0 = \langle 1,0,0\rangle$ is on the line $\Rightarrow$

$$x = 1+5t \qquad y = -2t \qquad z = -3t \qquad \text{or} \qquad \frac{x-5}{5} = -\frac{y}{2} = -\frac{z}{3}$$

eg. Where does the line $\underline{r} = \langle 1,0,0\rangle + t\langle 0,1,1\rangle$ intersect the plane $4x+2y=6$?

$4(1) + 2(t) = 6 \Rightarrow t=1$, so $\underline{r} = \langle 1,1,1\rangle$

• The distance from a point to a plane is found using the <u>scalar projection</u> of any vector from the point to the plane onto the normal to the plane. ie $|\text{comp}_{\underline{n}} \vec{PQ}| = \vec{PQ} \cdot \left(\frac{\underline{n}}{|\underline{n}|}\right)$ [Q can be any point on the plane].

need length of this vector; ie. perpendicular distance from P to the plane

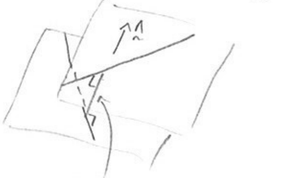
eg. find the distance from $P(1,0,0)$ to the plane $5x+y-z=1$

A point on the plane is $Q(0,1,0)$ and $\vec{PQ} = \langle -1,1,0\rangle$. The normal is $\underline{n} = \langle 5,1,-1\rangle$.
So the distance is $\left|\vec{PQ} \cdot \left(\frac{\underline{n}}{|\underline{n}|}\right)\right| = \left|\frac{-4}{\sqrt{27}}\right| = \frac{4}{3\sqrt{3}}$

eg. find the distance between the lines with equations $x=1+t \quad y=-2+3t \quad z=4-t$
$x=2s \quad y=3+s \quad z=-3+4s$

the lines are clearly not parallel (or their direction vectors would be multiples of each other)
do they intersect? If they did $\left.\begin{matrix} 1+t=2s \\ -2+3t=3+s \end{matrix}\right\} \Rightarrow 11-5t=0 \Rightarrow t=\frac{11}{5}, s=\frac{8}{5}$.
$4-t=-3+4s$ then $4-t=\frac{9}{5}$ & $-3+4s=\frac{17}{5}$ so this does not work $\Rightarrow$ no intersection

The lines are skew $\Rightarrow$ can be viewed as lying on parallel planes perpendicular to the common normal
$\underline{n} = \langle 1,3,-1\rangle \times \langle 2,1,4\rangle$ (ie the cross product of the direction vectors) $\Rightarrow \underline{n} = \langle 13,-6,-5\rangle$



If we take the scalar projection of <u>any</u> line connecting the two planes in direction $\underline{n}$, we find the required distance. eg. $P(1,-2,4)$ is on the first line, $Q(0,3,-3)$ is on the second. $\vec{PQ} = \langle -1,5,-7\rangle$, so the distance is

distance perpendicular to both lines
ie. in direction of normal to planes.

$$\left|\vec{PQ} \cdot \left(\frac{\underline{n}}{|\underline{n}|}\right)\right| = \left|\frac{-13-30+35}{\sqrt{169+25+36}}\right| = \frac{8}{\sqrt{230}} \approx 0.53$$