Cross product

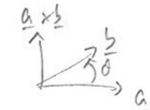
- If $\underline{a} = \langle a_1, a_2, a_3 \rangle$ and $\underline{b} = \langle b_1, b_2, b_3 \rangle$ the cross product of $\underline{a}$ and $\underline{b}$ is

$$\boxed{\underline{a} \times \underline{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle}$$

[This is also sometimes written $\underline{a} \wedge \underline{b}$ and called the 'wedge product']

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\underline{i} - (a_1 b_3 - a_3 b_1)\underline{j} + (a_1 b_2 - a_2 b_1)\underline{k}$$

3×3 determinant



The definition is chosen so that $\underline{a} \times \underline{b}$ is orthogonal to both $\underline{a}$ and $\underline{b}$, so

$\underline{a} \cdot (\underline{a} \times \underline{b}) = 0$
$\underline{b} \cdot (\underline{a} \times \underline{b}) = 0$ (Exercise: show this)

eg. What is $\langle 1,3,4 \rangle \times \langle 1,1,-2 \rangle$?

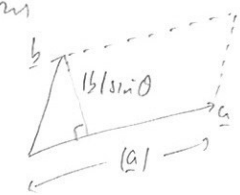
$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 4 \\ 1 & 1 & -2 \end{vmatrix} = (-6-4)\underline{i} - (-2-4)\underline{j} + (1-3)\underline{k} = \langle -10, 6, -2 \rangle$$

check orthogonal to $\langle 1,3,4 \rangle$: $\langle 1,3,4 \rangle \cdot \langle -10,6,-2 \rangle = 0$ ✓

- Right hand rule: if fingers curl from $\underline{a}$ to $\underline{b}$ through angle $< \pi$, then thumb points in direction $\underline{a} \times \underline{b}$.
- The cross product is related to the angle between the vectors

$$\boxed{|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta}$$

$$\boxed{\underline{a} \text{ and } \underline{b} \text{ are parallel} \iff \underline{a} \times \underline{b} = \underline{0}}$$

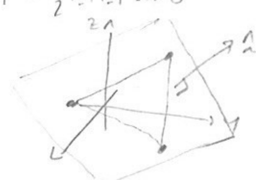


$|\underline{a} \times \underline{b}|$ is the area of the parallelogram formed by $\underline{a}$ and $\underline{b}$

The area of the triangle formed by $\underline{a}$ and $\underline{b}$ is

$\frac{1}{2}|\underline{a} \times \underline{b}| = \frac{1}{2}|\underline{a}||\underline{b}| \sin\theta$

eg. find a unit vector that is normal to the plane containing $P(1,0,1)$, $Q(0,2,2)$, $R(2,3,0)$



two vectors in the plane are $\vec{PQ} = \langle -1,2,1 \rangle$ and $\vec{PR} = \langle 1,3,-1 \rangle$

$$\underline{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix} = (-5)\underline{i} - (0)\underline{j} + (-5)\underline{k} = \langle -5, 0, 5 \rangle$$

A unit vector is $\hat{n} = \dfrac{\langle -5,0,5 \rangle}{\sqrt{50}} = \dfrac{1}{\sqrt{2}}\langle -1,0,1 \rangle$

eg. What is the area of the triangle with these points as vertices?

This is $\frac{1}{2}|\underline{a} \times \underline{b}|$ where $\underline{a}$ and $\underline{b}$ are two of the sides of the triangle, ie $\underline{a} = \vec{PQ}$, $\underline{b} = \vec{PR}$, then $\frac{1}{2}|\underline{a} \times \underline{b}| = \frac{\sqrt{50}}{2} = \frac{5}{2}\sqrt{2}$

- Note $\underline{i} \times \underline{i} = 0$, $\underline{i} \times \underline{j} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \underline{k}$, $\underline{i} \times \underline{k} = -\underline{j}$

[any vector cross itself must be 0 since they are parallel]

$\underline{i} \times \underline{j} \neq \underline{j} \times \underline{i}$ ⚠

Similarly, $\underline{j} \times \underline{i} = -\underline{k}$, $\underline{j} \times \underline{j} = \underline{0}$, $\underline{j} \times \underline{k} = \underline{i}$, $\underline{k} \times \underline{i} = \underline{j}$, $\underline{k} \times \underline{j} = -\underline{i}$, $\underline{k} \times \underline{k} = \underline{0}$

- Properties of the cross product

$\boxed{\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}}$

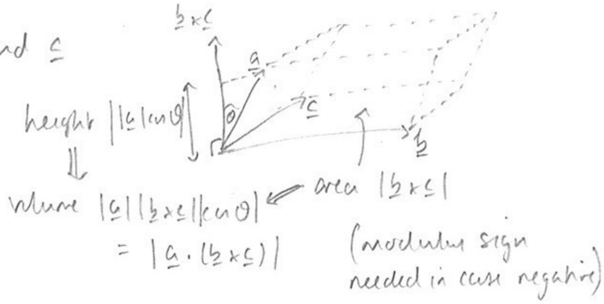
$(s\underline{a}) \times \underline{b} = s(\underline{a} \times \underline{b})$

$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$

$\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$ ← $\underline{a} \cdot (\underline{b} \times \underline{c})$ is the scalar triple product

$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$ ← $\underline{a} \times (\underline{b} \times \underline{c})$ is the vector triple product

- $|\underline{a} \cdot (\underline{b} \times \underline{c})|$ is the volume of the parallelepiped formed by $\underline{a}, \underline{b}$ and $\underline{c}$



- $\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$ if $\underline{a}, \underline{b}$ and $\underline{c}$ are co-planar

  eg. find condition on a so that $\underline{a} = \langle 1, 2, a \rangle$ is coplanar with $\underline{b} = \langle 1, 0, 1 \rangle$ and $\underline{c} = \langle 3, 1, 0 \rangle$

  $$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{vmatrix} = (-1)\underline{i} - (-3)\underline{j} + (1)\underline{k} = \langle -1, 3, 1 \rangle$$

  $$\underline{a} \cdot (\underline{b} \times \underline{c}) = -1 + 6 + a = 0 \text{ if } \underline{\underline{a = -5}}$$