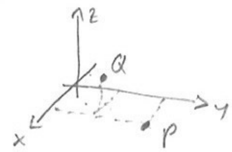
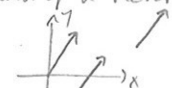
Coordinate Systems & Vectors

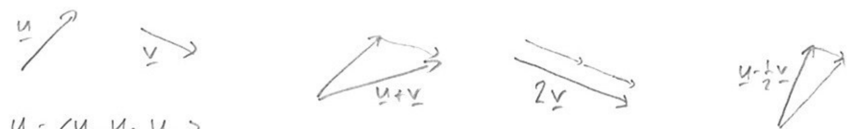
- A point P in $\mathbb{R}^3$ is represented by its coordinates relative to some origin O. The usual coordinates (x,y,z) are called <u>Cartesian coordinates</u> eg. $P(1,2,0)$ $Q(1,1,1)$



- The point can also be represented by a position <u>vector</u>, ie an arrow from the origin to the point: $\vec{p} = \underline{p} = \vec{OP} = \langle 1,2,0 \rangle$

[We use angled brackets to mean the components of a vector, and round brackets for the coordinates of a point. A vector could be drawn anywhere; eg.  all represent the vector $\langle 1,2 \rangle$. Only if the vector starts at the origin is it a position vector - then it has the same components as the coordinates of its end point]

- A vector is a quantity with a magnitude and direction. They can be added and subtracted, and multiplied by a scalar
  - Geometrically
  - Component-wise



$\underline{u} = \langle u_1, u_2, u_3 \rangle$
$\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\quad$ $\underline{u}+\underline{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3 \rangle$ $\quad$ $s\underline{u} = \langle su_1, su_2, su_3 \rangle$

- The magnitude of a vector is $\boxed{|\underline{v}| = \sqrt{v_1^2+v_2^2+v_3^2}}$ $\quad$ $|\underline{v}|$ is sometimes written as $\|\underline{v}\|$ and is also called the <u>norm</u> or <u>modulus</u> of the vector.

eg. Find the vector from Q to P, and the distance between them

$$\vec{QP} = \vec{OP} - \vec{OQ} = \langle 1,2,0 \rangle - \langle 1,1,1 \rangle = \langle 0,1,-1 \rangle \qquad |\vec{QP}| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

eg. A sphere with centre at $C(h,k,l)$ and radius r is described by the equation

$|\underline{r} - \underline{c}| = r$, where $\underline{r} = \langle x,y,z \rangle$ is the position vector of a point on the sphere
$\underline{c} = \langle h,k,l \rangle$ is the position vector of the centre point

ie. the sphere is all the points that are a distance r away from the centre.

In component form, this equation is $\boxed{(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2}$

- A <u>unit vector</u> has length 1. Any vector can be converted to a unit vector by dividing by its length $\frac{\underline{v}}{|\underline{v}|}$; this represents the <u>direction</u> of the vector.

The <u>standard basis vectors</u> are the unit vectors in the x, y and z directions $\boxed{\underline{i} = \langle 1,0,0 \rangle \quad \underline{j} = \langle 0,1,0 \rangle \quad \underline{k} = \langle 0,0,1 \rangle}$

Vectors are sometimes written in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$: eg. $\underline{v} = \langle 1,2,0 \rangle$ can be written $\underline{v} = \underline{i} + 2\underline{j}$

- General properties of vectors (ie requirements for a vector space), if $\underline{a}, \underline{b}, \underline{c}$ are vectors, s,t are scalars

$\underline{a}+\underline{b} = \underline{b}+\underline{a}$ $\qquad$ $\underline{a}+(\underline{b}+\underline{c}) = (\underline{a}+\underline{b})+\underline{c}$

$\underline{a}+\underline{0} = \underline{a}$ $\qquad$ $\underline{a}+(-\underline{a}) = \underline{0}$

$s(\underline{a}+\underline{b}) = s\underline{a}+s\underline{b}$ $\qquad$ $(s+t)\underline{a} = s\underline{a}+t\underline{a}$

$(st)\underline{a} = s(t\underline{a})$ $\qquad$ $1\underline{a} = \underline{a}$

Dot product

- If $\underline{a} = \langle a_1, a_2, a_3 \rangle$, $\underline{b} = \langle b_1, b_2, b_3 \rangle$, the dot product of $\underline{a}$ and $\underline{b}$ is $\boxed{\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$

  It gives a scaler, and is sometimes called the scaler product or inner product

  eg. $\langle 2,4 \rangle \cdot \langle 1,-1 \rangle = 2 - 4 = -2$

  $\langle 1,1,2 \rangle \cdot \langle \frac{1}{2}, 1, 0 \rangle = \frac{1}{2} + 1 + 0 = \frac{3}{2}$

- Properties of the dot product

  $\underline{a} \cdot \underline{a} = |\underline{a}|^2$ $\qquad \underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

  $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ $\qquad (s\underline{a}) \cdot \underline{b} = s(\underline{a} \cdot \underline{b})$

  $\underline{a} \cdot \underline{0} = 0$

- The dot product is related to the angle between the vectors $\boxed{\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta}$

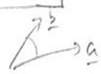
  [This can be shown using the cosine formula for a triangle $|\underline{a} - \underline{b}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}| \cos\theta$]

  eg. What is the angle between the vectors $\langle 1,2,0 \rangle$ and $\langle 1,1,3 \rangle$?

  $\cos\theta = \frac{1+2+0}{\sqrt{5}\sqrt{11}} = \frac{3}{\sqrt{55}}$ $\qquad \theta = \cos^{-1}\left(\frac{3}{\sqrt{55}}\right)$

- $\boxed{\underline{a} \text{ and } \underline{b} \text{ are orthogonal (perpendicular)} \Leftrightarrow \underline{a} \cdot \underline{b} = 0}$ $\qquad (\theta = \frac{\pi}{2})$

  Also note $\underline{a} \cdot \underline{b} > 0$ if $0 < \theta < \frac{\pi}{2}$



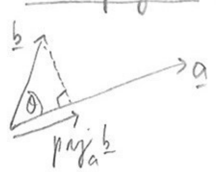
  $\underline{a} \cdot \underline{b} < 0$ if $\frac{\pi}{2} < \theta < \pi$

- The 'direction angles' of a vector are the angles with the x, y and z directions

  If $\underline{a} = \langle a_1, a_2, a_3 \rangle$ then $\cos\alpha = \frac{\underline{a} \cdot \hat{i}}{|\underline{a}|} = \frac{a_1}{|\underline{a}|}$, $\cos\beta = \frac{a_2}{|\underline{a}|}$, $\cos\gamma = \frac{a_3}{|\underline{a}|}$

Projection

- The projection of $\underline{b}$ onto $\underline{a}$ is the 'part of $\underline{b}$ in direction of $\underline{a}$'. It is denoted $\text{proj}_{\underline{a}} \underline{b}$. Its length is the scaler projection, denoted $\text{comp}_{\underline{a}} \underline{b}$.



  By geometry $\boxed{\text{comp}_{\underline{a}} \underline{b} = |\underline{b}| \cos\theta = |\underline{b}| \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \underline{b} \cdot \left(\frac{\underline{a}}{|\underline{a}|}\right)}$ ie. the dot product with unit vector in direction of $\underline{a}$

  To get the projection, multiply by the unit vector in the direction of $\underline{a}$, so

  $$\boxed{\text{proj}_{\underline{a}} \underline{b} = \left(\underline{b} \cdot \left(\frac{\underline{a}}{|\underline{a}|}\right)\right) \frac{\underline{a}}{|\underline{a}|}}$$

  eg. find the projection of $\underline{b} = \langle 1,1,2 \rangle$ onto $\underline{a} = \langle -2,3,1 \rangle$

  $\text{comp}_{\underline{a}} \underline{b} = \frac{\underline{b} \cdot \underline{a}}{|\underline{a}|} = \frac{-2+3+2}{\sqrt{14}} = \frac{3}{\sqrt{14}}$ $\qquad \text{proj}_{\underline{a}} \underline{b} = \frac{3}{\sqrt{14}} \frac{\langle -2,3,1 \rangle}{\sqrt{14}} = \frac{3}{14} \langle -2,3,1 \rangle$

- The orthogonal projection of $\underline{b}$ onto $\underline{a}$ is $\underline{b} - \text{proj}_{\underline{a}} \underline{b}$