Here are solution hints for practice exam:

Multiple choice questions: A, C, E, A, C, D, D, E, D, D.

Long answer questions (partial hints:)

Q 11 :

$$-x^2 + x + \frac{2}{13}\ln|x+5| + \frac{11}{13}\ln|x-8| + C$$

Q 12: Equilibria: $-2$ unstable, 0 stable, 3 unstable

Q 13: The general solution is $(2,-1,0) + t(-1,2/3,1), t \in \mathbb{R}$.

Q 14: Domain $\{(x,y) : y > 0\} \subset \mathbb{R}^2$, Range $\mathbb{R}$. Level curve for $k \in \mathbb{R}$: $y = e^{k+x}, x \in \mathbb{R}$. Tangent plane $z = -x + \frac{1}{e}y$.

Q 15:

$$\frac{2}{85} - \frac{9}{85}i$$

Q 16: Eigenvalues and eigenvectors: $\lambda_1 = -4, v_1 = s(1,-1), s \neq 0$, $\lambda_2 = -2, v_1 = t(1,1), t \neq 0$. (The conditions $s, t \neq 0$ are required!). The equilibrium is $(0,0)$ and it is stable. The $x$-nullcline is the line $y = 3x$, the $y$-nullcline is the line $y = 1/3x$.

Q 17: Equilibria: $(4,0)$ ( not stable, eigenvalues of Jacobian are $-3$ and 6 ), $(2,1)$ (stable, eigenvalues of Jacobian are $-3 \pm 3i$) Trajectories that start on the $x$-axis should approach the equilibrium $(4,0)$, all other trajectories should approach the the equiuilibrium $(2,1)$. See posted review for more details on this type of problem.

$\boxed{Q1}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{(\sin^2(x))^{\frac{1}{3}}} = $ $\boxed{\begin{array}{l} \sin x = u \\ \cos x\,dx = du \end{array} \quad \begin{array}{c|c|c} x & 0 & \frac{\pi}{2} \\ \hline u & 0 & 1 \end{array}}$

$$= \int_0^1 \frac{du}{u^{\frac{2}{3}}} \overset{\text{type 2}}{=} \lim_{\varepsilon \to 0} \int_\varepsilon^1 u^{-\frac{2}{3}}\,du = \lim_{\varepsilon \to 0} 3u^{\frac{1}{3}}\Big|_\varepsilon^1 =$$

$$= \lim_{\varepsilon \to 0} \left[3(1)^{\frac{1}{3}} - 3\varepsilon^{\frac{1}{3}}\right] = 3$$

A

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$\boxed{Q2}$ $\displaystyle\int_1^e x^2 \ln(x^3)\,dx = \frac{x^3}{3}\ln(x^3)\Big|_1^e -$

$$-\int_1^e \frac{x^3}{3}\,\frac{3x^2}{x^3}\,dx = \frac{x^3}{3}\cdot 3\ln x\Big|_1^e - \int_1^e x^2\,dx =$$

$$= e^3\ln e - 1\cdot\ln 1 - \frac{e^3}{3} + \frac{1}{3} = \frac{2e^3}{3} + \frac{1}{3}$$

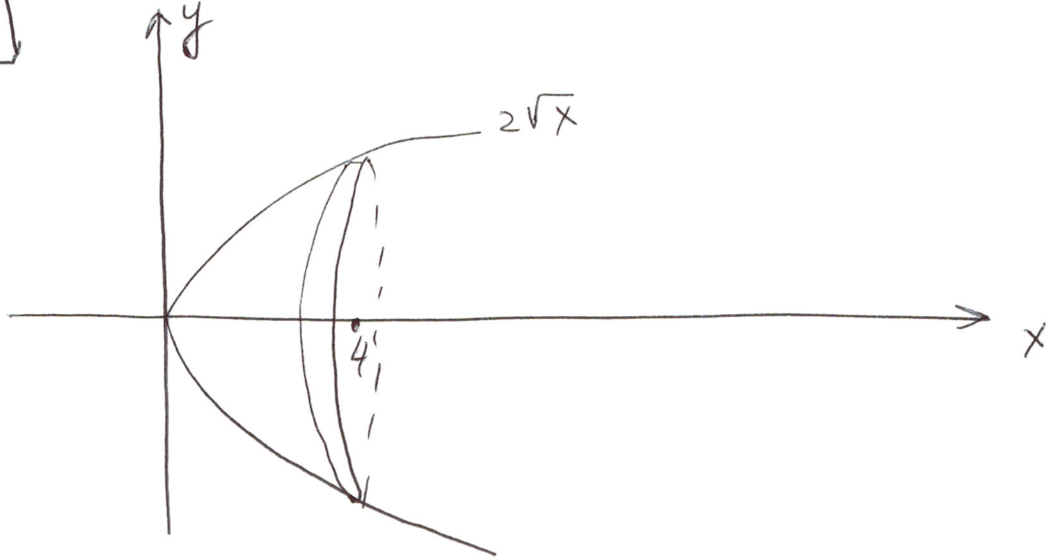
C

Here, I used integration by parts method:

$\int u\,dv = uv - \int v\,du$

$$\boxed{\begin{array}{ll} x^2dx = dv & \ln(x^3) = u \\ \frac{x^3}{3} = v & \frac{3x^2}{x^3}dx = \frac{3}{x}dx = du. \end{array}}$$

$\boxed{Q3}$





$$A_i = \pi r^2 = \pi \cdot 4 \cdot x$$

$$V_x = \pi \cdot 4 \cdot x\, dx$$

$$V = 4\pi \int_0^4 x\, dx = 32\pi$$

E

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$\boxed{Q4}$

$$\frac{dx}{dt} = \frac{3x}{t^2}$$

$$x(1) = e$$

$$x(3) - ?$$

$$\frac{dx}{x} = \frac{3dt}{t^2}$$

$$\ln|x| = -\frac{3}{t} + c$$

$$|x| = e^{-\frac{3}{t}+c} = c \cdot e^{-\frac{3}{t}}$$

$$x(t) = \pm c e^{-\frac{3}{t}} = \hat{c} \cdot e^{-\frac{3}{t}}$$

$$x(1) = \hat{c}\, e^{-3} = e \quad \Rightarrow \quad \hat{c} = e^4$$

$$x(t) = e^4 \cdot e^{-\frac{3}{t}}$$

$$x(3) = e^4 \cdot e^{-1} = e^3$$

A

Q5 $X = \begin{bmatrix} 2 & -1 & 7 \\ 8 & 6 & -2 \end{bmatrix}$ , $Y = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$ ,

$$Z = \begin{bmatrix} 3 & -1 & 0 \\ 5 & 1 & 0 \\ 7 & 7 & 1 \end{bmatrix}$$

(i) $XY$ is defined

$$\begin{bmatrix} 2 & -1 & 7 \\ 8 & 6 & -2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \quad \boxed{\text{No}}$$

$(2\times 3) \times (2\times 2)$

(ii) $XZ$ is defined $\boxed{\text{Yes}}$

$$\begin{bmatrix} 2 & -1 & 7 \\ 8 & 6 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 5 & 1 & 0 \\ 7 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[2 \times \underline{\underline{3}}] \times [\underline{\underline{3}} \times 3] = [2 \times 3]$$

(iii) $YX^T$ is defined

$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ -1 & 6 \\ 7 & -2 \end{bmatrix} \quad \boxed{\text{No}}$$

$(2\times \underline{2}) \times (\underline{3}\times 2)$

(IV) $ZX^T$ is defined $\boxed{\text{Yes}}$

$$\begin{bmatrix} 3 & -1 & 0 \\ 5 & 1 & 0 \\ 7 & 7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 8 \\ -1 & 6 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$[3\times \underline{\underline{3}}]$ $[\underline{\underline{3}} \times 2]$ $[3\times 2]$

(V) The matrix $Y$ is invertible.

$$\det(Y) = 4 \cdot 4 - 8 \cdot 2 = 0$$

$\boxed{\text{No, it is not.}}$

(VI) $Z + X^T X$ is defined

$(3\times3) + (3\times2)\cdot(2\times3)$

$\boxed{\text{Yes}}$, $Z + X^T X$ is defined.

The answer is $\boxed{C}$

$\boxed{Q6}$ $\begin{bmatrix} -\sqrt{3} & 7 & 7 \\ -4 & \sqrt{3} & 0 \\ 0 & 0 & -7 \end{bmatrix}$

$A\bar{v} = \lambda\bar{v}$ $\quad$ $(A-\lambda I)$ does not have the inverse,

$(A-\lambda I)\bar{v} = 0$ $\quad$ i.e $\det(A-\lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} -\sqrt{3}-\lambda & 7 & 7 \\ -4 & \sqrt{3}-\lambda & 0 \\ 0 & 0 & -7-\lambda \end{vmatrix} =$$

$$= (-\sqrt{3}-\lambda)(\sqrt{3}-\lambda)(-7-\lambda) + 0 + 0 - 0 - 0 + 4\cdot 7(-7-\lambda) = 0$$

$$= -(\sqrt{3}+\lambda)(\sqrt{3}-\lambda)(-1)(7+\lambda) - 28(7+\lambda) = 0$$

$$(\sqrt{3}+\lambda)(\sqrt{3}-\lambda)(7+\lambda) - 28(7+\lambda) = 0$$

$7+\lambda = 0$ $\quad$ $(\sqrt{3}+\lambda)(\sqrt{3}-\lambda) - 28 = 0$

$\boxed{\lambda_1 = -7}$ $\quad$ $3 - \lambda^2 - 28 = 0$

$\lambda^2 = -25$

$\boxed{\lambda_2 = 5i, \ \lambda_3 = -5i}$

$\boxed{D}$ $\lambda = 7$ is $\boxed{\text{not}}$ an eigenvalue.

$\boxed{Q7}$ $z = -3 + 3i$

(i) $|z| = r = \sqrt{(-3)^2 + (3)^2} = \sqrt{18} = 3\sqrt{2}$ $\boxed{\text{Yes}}$

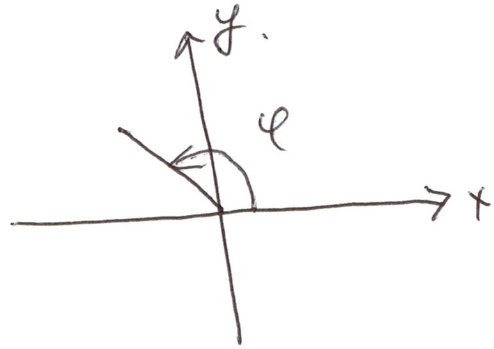
(ii) $x = -3$ $\quad x = r \cdot \cos\varphi$

$y = 3$ $\quad y = r \sin\varphi$

$\frac{y}{x} = -1 = \tan\varphi.$

$\varphi = \arctan(-1) + \pi$

$\varphi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}.$




Assume $\varphi \in [-\pi, \pi)$!

or $\quad -3 = 3\sqrt{2}\cos\varphi \quad \Rightarrow \quad \left.\begin{array}{l} \cos\varphi = -\frac{1}{\sqrt{2}} \\ \sin\varphi = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \varphi = \frac{3\pi}{4}.$

$\quad 3 = 3\sqrt{2}\sin\varphi$

$z = r \cdot e^{i\varphi} = r(\cos\varphi + i\sin\varphi) = \underbrace{3\sqrt{2}}_{\text{is missing in (ii)}} e^{i\frac{3\pi}{4}}$

(ii) $\boxed{\text{No}}$

(iii) $z = |z|\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(iii) $\boxed{\text{No}}$

(iv) $\boxed{\text{Yes}}$

(v) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ is always true $\boxed{\text{Yes}}$

$\boxed{D}$

$\boxed{Q8}$ $f(x,y) = \begin{pmatrix} x^2 + x + y \\ yx + x^2 \end{pmatrix}$ at $(1,2)$

Jacobian matrix $= \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(1,2)} = \begin{pmatrix} 2x+1 & 1 \\ y+2x & x \end{pmatrix}_{(1,2)} =$

$= \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ $\boxed{E}$

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$\boxed{Q9}$ $\frac{dx}{dt} = x^2 - 2xy + y^2 = f(x,y)$

$\frac{dy}{dt} = e^{x^2-1} - 1 - \ln y = g(x,y)$

$f(x,y) = 0$ $\quad x^2 - 2xy + y^2 = 0$ (1)

$g(x,y) = 0$ $\quad e^{x^2-1} - 1 - \ln y = 0.$ (2)

A: $(x,y) = (0,2)$ $\quad$ (1) x

B: $(x,y) = (5,2)$ $\quad$ (1) $\Rightarrow 25 - 2\cdot5\cdot2 + 4 \overset{?}{=} 0$ $\quad 9 \neq 0$

C: $(x,y) = (3,3)$ $\quad 9 - 2\cdot9 + 9 \overset{?}{=} 0$ Yes. $f(3,3) = 0$

$g(3,3) \overset{?}{=} 0$ $\quad e^8 - 1 - \ln 3 \overset{?}{=} 0$

$\boxed{D: (x,y) = (1,1)}$ $\quad f(1,1) = 0$

$g(1,1) = e^0 - 1 - \ln 1 = 1 - 1 - 0 = 0.$ ✓

E: $(x,y) = (1,3)$ $\quad f(1,3) = 1 - 6 + 9 \neq 0.$

$\boxed{Q10}$ $z = f(x,y) = \pi - 2\cos 3x + 4\sin 5y$

$(x_0, y_0) = (0, \pi)$

$z = z_0 + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$z_0 = f(x_0,y_0) = f(0,\pi) = \pi - 2\cos 0 + 4\sin 5\pi =$

$= \pi - 2.$

$f_x = -2(\cos 3x)_x + 0 = -2 \cdot (-\sin 3x) \cdot 3 =$

$= 6\sin 3x$

$f_x(0,\pi) = 0.$

$f_y = 0 - 0 + 4(\sin 5y)'_y = 4 \cdot 5\cos 5y = 20\cos 5y$

$f_y(0,\pi) = 20\cos 5\pi = -20.$

$z = \pi - 2 + 0 - 20(y - \pi) = \pi - 2 - 20y + 20\pi =$

$= -20y + 21\pi - 2.$

$\boxed{D}$

$\boxed{Q11}$ $I=\displaystyle\int \frac{-2x^3+7x^2+78x-37}{x^2-3x-40}\,dx$

$$
\begin{array}{r|l}
 & -2x+1 \\
x^2-3x-40 & -2x^3+7x^2+78x-37 \\
- & -2x^3+6x^2+80x \quad \downarrow \\
 & \quad x^2-2x-37 \\
- & \quad x^2-3x-40 \\
 & \qquad x+3
\end{array}
$$

Thus, $(x^2-3x-40)\cdot(-2x+1)+(x+3)=-2x^3+7x^2+78x-37$

$$I=\underbrace{\int(-2x+1)\,dx}_{I_1}+\underbrace{\int\frac{(x+3)\,dx}{x^2-3x-40}}_{I_2.}$$

$$I_1=-\frac{2x^2}{2}+x+C=-x^2+x+C$$

$$I_2=\int\frac{(x+3)\,dx}{x^2-3x-40}$$

$x^2-3x-40=0$

$\Delta=9-4\cdot(-40)=169>0$

$x_1=8,\quad x_2=-5$

Thus, $\dfrac{(x+3)}{(x-8)(x+5)}=\dfrac{A}{x-8}+\dfrac{B}{x+5}$

$A(x+5)+B(x-8)=x+3$

$\begin{cases} A+B=1 \\ 5A-8B=3 \end{cases} \Rightarrow A=\dfrac{11}{13},\ B=\dfrac{2}{13}$

$$I_2 = \frac{11}{13}\int \frac{dx}{x-8} + \frac{2}{13}\int \frac{dx}{x+5} =$$

$$= \frac{11}{13}\ln|x-8| + \frac{2}{13}|x+5| + C.$$

$$I = -x^2 + x + \frac{11}{13}\ln|x-8| + \frac{2}{13}\ln|x+5| + C.$$

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$\boxed{Q12}$ $x' = \dfrac{dx}{dt} = x^3 - x^2 - 6x = f(x)$

(a) Equilibria: $f(x) = 0$

$$x_1^* = 0$$
$$x_2^* = 3$$
$$x_3^* = -2.$$

(b) Use The Stability Criterion for Autonomous dif. eq-ns.

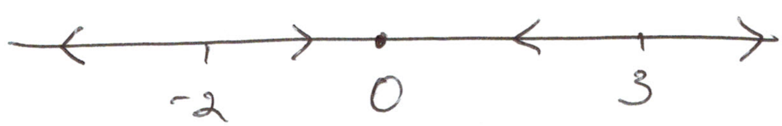
$f'(x) = 3x^2 - 2x - 6$

$f'(0) = -6 < 0 \Rightarrow x_1^* = 0$ is stable.

$f'(3) > 0 \Rightarrow x_2^*$ is unstable

$f'(-2) > 0$, $x_3^*$ is unstable.

(c)



$\boxed{Q13}$

$$\begin{cases} 2x-3y+4z=7 \\ -5x-12y+3z=2 \\ -3x-15y+7z=9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 7 \\ -5 & -12 & 3 & 2 \\ -3 & -15 & 7 & 9 \end{array}\right] \begin{array}{l} R1 \to R1 \\ R2 \to R2+R1 \\ R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 2 & -3 & 4 & 7 \\ -3 & -15 & 7 & 9 \\ -3 & -15 & 7 & 9 \end{array}\right] \to$$

$$\to \begin{array}{l} R1 \to \frac{R1}{2} \\ R2 \to -\frac{R2}{3} \\ R3 \to R3-R2 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{7}{2} \\ 1 & 5 & -\frac{7}{3} & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R2 \to R2-R1]{} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{7}{2} \\ 0 & 5+\frac{3}{2} & -\frac{7}{3}-2 & -3-\frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{7}{2} \\ 0 & \frac{13}{2} & -\frac{13}{3} & -\frac{13}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R2 \to \frac{R2 \cdot 2}{13}]{} \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & \frac{7}{2} \\ 0 & 1 & -\frac{2}{3} & -1 \\ 0 & 0 & 0 & \end{array}\right]$$

row echelon form.

$z = t, \ t \in \mathbb{R}.$

$(2) \Rightarrow \quad y - \frac{2}{3}z = -1$

$\qquad y = \frac{2}{3}z - 1 = \frac{2}{3}t - 1$

$(1) \Rightarrow \quad x - \frac{3}{2}y + 2z = \frac{7}{2}$

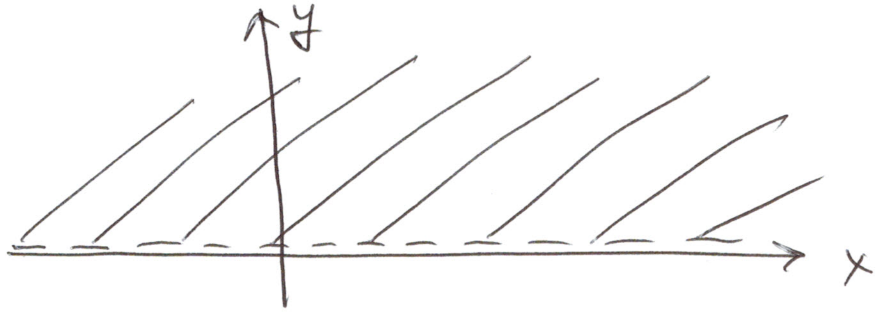
$\qquad x = \frac{3}{2}y + \frac{7}{2} - 2z = \frac{3}{2}\left(\frac{2}{3}t - 1\right) - 2t + \frac{7}{2} =$

$\qquad = 2 - t.$

$$\boxed{x = 2-t, \ y = \frac{2}{3}t - 1, \ z = t}$$

Q14 (a,b) $z = \ln y - x$

The domain: $y > 0$, $x \in (-\infty, +\infty)$




Range $z \in (-\infty, +\infty)$

(c) $f(x,y) = C$

$\ln y - x = C$

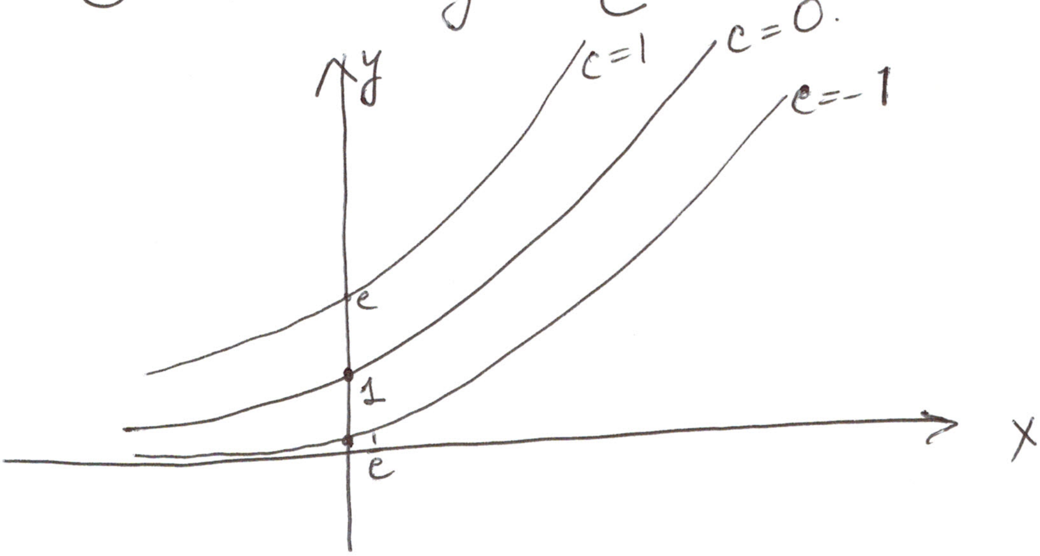
$\ln y = C + x$

$y = e^{C+x} = e^{C} \cdot e^{x}$

$C = 0$. $\quad y = e^{x}$

$C = 1 \quad y = e \cdot e^{x}$

$C = -1 \quad y = \dfrac{e^{x}}{e}$



(d) $L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} (x-x_0) +$

$+ \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} (y-y_0) \qquad (x_0,y_0) = (2,e)$

$z_0 = f(x_0,y_0) = f(2,e) = \ln e - 2 = -1$

$f_x = -1 \qquad f_x(x,y) = -1.$

$f_x(2,e) = -1.$

$f_y = \frac{\partial f}{\partial y} = \frac{1}{y}. \qquad f_y(x,y) = \frac{1}{y}$

$f_y(2,e) = \frac{1}{e}.$

$L(x,y) = -1 - 1(x-2) + \frac{1}{e}(y-e) =$

$= -1 - x + 2 + \frac{1}{e}y - 1 = \frac{y}{e} - x$

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$\boxed{Q\,15}$ $\omega = 3-4i \quad , z = -6+7i$

$\frac{|\omega|}{\omega z} = \frac{\sqrt{(3)^2+(-4)^2}}{(3-4i)(-6+7i)} = \frac{5}{-18+24i+21i-28i^2} =$

$= \frac{5}{-18+45i+28} = \frac{5}{10+45i} = \frac{1}{2+9i} =$

$\frac{1\cdot(2-9i)}{(2+9i)(2-9i)} = \frac{2-9i}{4+81} = \frac{2}{85} - \frac{9i}{85}$

$\boxed{Q16}$

$$\frac{dx}{dt} = -3x + y$$
$$\frac{dy}{dt} = x - 3y$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} =$$

$$= (-3-\lambda)(-3-\lambda) - 1 = (3+\lambda)^2 - 1 = \lambda^2 + 6\lambda + 8 = 0$$

$$\lambda_1 = -2; \quad \lambda_2 = -4.$$

$\boxed{\lambda_1 = -2}$ $\qquad A\bar{v}^1 = \lambda_1 \bar{v}^1$

$$(A - \lambda_1 I)\bar{v}^1 = \bar{0}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$v_2^1 = t$$
$$v_1^1 = t \quad , t \in \mathbb{R}, \; t \neq 0.$$

~~taken~~ $\bar{v}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t.$

$\boxed{\lambda_2 = -4}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t, \ t \in \mathbb{R}.$

The general solution has the following form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The Equilibrium.

$\frac{dx}{dt} = 0$ $(x^*, y^*) = (0, 0)$

$\frac{dy}{dt} = 0.$

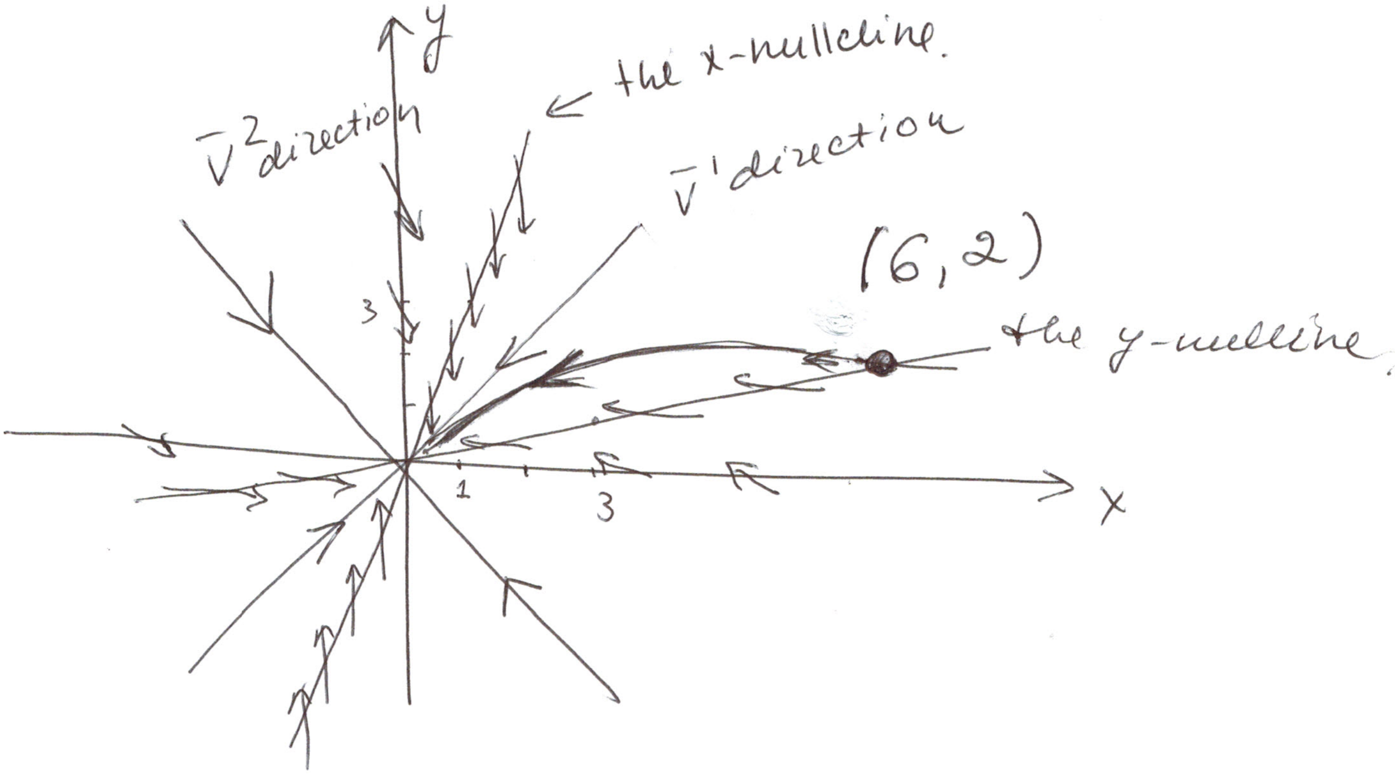
The system is linear, thus the only equilibrium we have is the origin (see lecture notes)

Both eigenvalues are negative, the equilibrium is a stable node

the x-nullcline. $-3x + y = 0$

$\boxed{y = 3x}$

the y-nullcline $x - 3y = 0$ $\boxed{y = \frac{x}{3}}$




the x-nullcline

$y = 3x$

$\frac{dy}{dt} = x - 3y = x - 9x = -8x.$

$\frac{dy}{dt} < 0 \quad x > 0 \quad y \downarrow.$

$x < 0 \quad y \uparrow.$

$\boxed{Q17}$

$$\frac{dx}{dt} = 12 - 3xy - 3x$$
$$\frac{dy}{dt} = 3xy - 6y$$

(a) Equilibria $\quad \frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0.$

$$\begin{cases} 12 - 3xy - 3x = 0 & \Leftrightarrow \quad 4 - xy - x = 0 \\ 3xy - 6y = 0 & \Leftrightarrow \quad y(x-2) = 0 \end{cases}$$

$y = 0. \quad x = 4.$

$x = 2. \quad 4 - 2y - 2 = 0$
$\quad y = 1.$

$$\boxed{\begin{aligned}(x_1^*, y_1^*) &= (4,0) \\ (x_2^*, y_2^*) &= (2,1)\end{aligned}}$$

(b) $$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -3y-3 & -3x \\ 3y & 3x-6. \end{bmatrix}$$

(c) $$J(4,0) = \begin{bmatrix} -3 & -12 \\ 0 & 6 \end{bmatrix}$$

$\lambda_1 = -3 \; ; \; \lambda_2 = 6.$

$\bar{v}^2 = \begin{bmatrix} \frac{4}{3} \\ 1 \end{bmatrix} \quad \bar{v}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(x_1^*, y_1^*) = (4,0)$ is a saddle point (unstable).

$$J(2,1) = \begin{bmatrix} -6 & -6 \\ 3 & 0 \end{bmatrix} = B$$

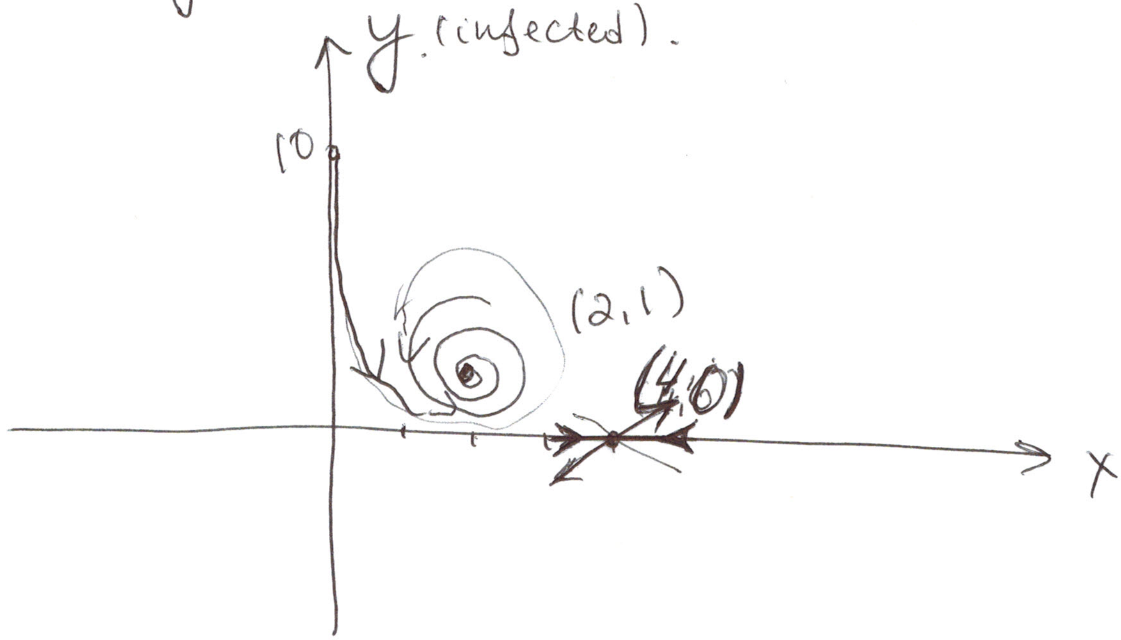
$$\det(B - \lambda I) = (-6-\lambda)\cdot(-\lambda) + 18 = \lambda^2 + 6\lambda + 18 = 0$$

$\lambda_1 = -3+3i$

$\lambda_2 = -3-3i$

$Re(\lambda_1, \lambda_2) = -3 < 0 \implies$

$(x_2^*, y_2^*)$ is a stable spiral.




(f) if 10 people are initially infected, then in the long term the population will stabilize at $(2,1)$.

We will have 1 infected and 2 susceptible individuals.