L3/L4

Chapters 4,5 Vector Spaces

(examples from 1.1)

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ – geometric vectors

$\mathbb{R}^n$, $n \geq 4$ "vectors"

NON-geometric, NON-"$\mathbb{R}^n$" vectors . . .

e.g. "Spaces" of equations.

$E_1: \; x - y - z = 1$

$E_2: \; 2x - y + z = 1$

$E_3: \; -x + 2y + 4z = 4$

$E_2 - 2E_1 = E_4: y + 3z = 3$

$E_1 + E_3: \; y + 3z = 3$

; $E_5: \; E_1 + E_4: x + 2z = 2$

We can write: $E_1 + E_3 = E_2 - 2E_1$

$3E_1 - E_2 + E_3 = 0$? •••

Remarks: 1) can add eqns to get another eqn

2) can multiply " by a scalar to get another eqn

3) equations have negatives!

$-E_1: \; -x - y + z = 1$

$E_1 + (-E_1) = \; ?$ "zero equation"

4) "zero equation" $0 = 0$ works!

5) Certain usual arithmetic holds:

$E_1 + E_2 = E_2 + E_1$

$E_1 + (E_2 + E_3) = (E_1 + E_2) + E_3$

$k(E_1 + E_2) = kE_1 + kE_2 \qquad k \in \mathbb{R}$

⋮

There are exactly the (10) properties of $\mathbb{R}^n$ (Thm 1.1., P.4) !

One could consider the space $\mathcal{E}$ of all equations "obtainable from" generated by E_1 & E_2 & E_3. Note that $\mathcal{E}$ behaves exactly like a space of vectors! "linear combinations of equations"

$\mathcal{E} = \{k_1E_1 + k_2E + k_3E_3 \mid k_i \in \mathbb{R}\}$!

(Q: is there an eqn of the form $y = y_0$ ($x = x_0$?) in $\mathcal{E}$?

i.e. can we solve for y?

i.e. Can we find $y_0, a, b \in \mathbb{R}$ s.t.

$(y = y_0)$ is $aE_1 + bE_2$?

exercise: no

- e.g. Formation of HBr (Stoichiometry)

— see web notes P1, 2.

(H

"vectors" don't have to be geometric, or even "n-tuples" (3)

1.1 What do we really need here?

V-set of vectors of some sort; operations: "add^n" of vectors, "mult^n" of vectors by scalar

"Closure":
- the sum of 2 vectors should be a vector; $u, v \in V \Rightarrow u+v \in$
- the scalar multiple of a vector should be a vector $a \in \mathbb{R},\ v \in V \Rightarrow a.v \in V$

"Existence":
- there should be a zero vector 0; $0 + v = v$
- every vector should have a negative given $v \in V$, there is $-v \in V$ s.t. $v + (-v) = 0$ (prevents $-v = -1.v$?

- Arithmetic properties: $v + w = w + v$
  $u + (v + w) = (u + v) + w$
  $a(v + w) = av + aw$
  $(a + b)v = av + bv$
  $a(bv) = (ab)v$
  $1.v = v$

Any set V, with 2 operations as above satisfying these (10!) these conditions "axioms" is called a <u>VECTOR SPACE</u>

(See P293, (5.1))

(4) e.g. $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{C}$ with usual ops.

eg $V = \{(x, 2x) \mid x \in \mathbb{R}\}$, standard ops from $\mathbb{R}^2$

C ✓ E ✓ A — ok, all works for $\mathbb{R}^2$ ✓

eg $U = \{(x, x+2) \mid x \in \mathbb{R}\}$; usual ops of $\mathbb{R}^2$

Usual ops $\Rightarrow$ usual zero $= (0,0) \notin U$. $\therefore$ ✗

eg $M_{22}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

Add^n, mult^n by Scalars : componentwise

C ✓ E ✓ A (like $\mathbb{R}^4$, stacked).

($M_{mn}(\mathbb{R}) = \{ A \mid A$ is an $m \times n$ matrix with real entries$\}$)

eg <u>Spaces of functions</u> (#5, P.295)

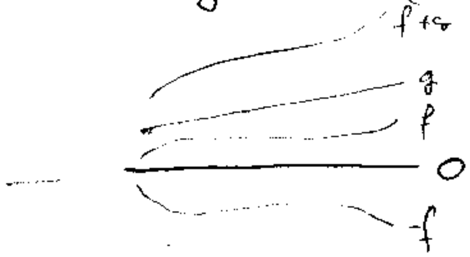
$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

$F[a,b] = \{ f \mid f : [a,b] \to \mathbb{R} \}$ N.B. $f = g \Leftrightarrow f(x) = g(x)$ $\forall x \in [a,b]$

$f, g \in F[a,b]$; $k \in \mathbb{R}$

<u>Add</u>^{n} $(f+g)(x) = f(x) + g(x)$; $f+g \in F[a,b]$

mult^n by scalar $(kf)(x) = k.f(x)$; $kf \in F[a,b]$



C ✓

E $0(x) = 0$, $\forall x \in [a,b]$

$(-f)(x) = -f(x)$

A ✓

- Can also consider

$F(\mathbb{R}) = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$, same ops ("ptwise")

$f(x) = \cos x$, $f \in F(\mathbb{R})$

$g(x) = x + x^2$, $g \in F(\mathbb{R})$, all polynomial function

$\left.\begin{array}{l} h(x) = \frac{1}{x} \\ t(x) = \tan x \end{array}\right\} \notin F(\mathbb{R})$; not def'd everywhere.

5.1.2 Subspaces & Spanning Sets

Suppose V is a v.s. and $W \subset V$ is a subset of V

(eg $\{(x, 2x) \mid x \in \mathbb{R}\}$). Give W same ops as V.

Is W a v.s.?

C – add^n ?
– mult by scalars ?

E – zero ?
– -ves $-1 \cdot v = -v$ ...
(true from axioms! Thm 2, P 296)

A – true in V, so true in W

Def^n A subset W of a v.s V is a subspace of V if W is a v.s. when given the same ops as V.

e.g. $\{(x, 2x) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ | e.g. $\{0\}$ is a s.s of V; V " " of V

Thm 3 (Subspace Test) If V is a v.s. and $W \subset V$, then
P. 297 W is a subspace of V iff

SHORTCUT
1. $0 \in W$
2. W is closed under add^n
3. " mult^n by scalars.

eg $T = \{ v \in \mathbb{R}^3 \mid v \cdot (1,-2,1) = 0 \} \subset \mathbb{R}^3$
$= \{ (x,y,z) \mid x - 2y + z = 0 \}$ $(\neq \mathbb{R}^3)$

<u>Subspace Test</u>

1) $\underset{(0,0,0)}{0} \in T$ since $(0,0,0) \cdot (1,-2,1) = 0$

2) If $u, v \in T$, then

$$(u+v) \cdot (1,-2,1) = u \cdot (1,-2,1) + v(1,-2,1) = 0 + 0 = 0$$

$\therefore u+v \in T$ $\therefore T$ closed under $+$

3) If $u \in T$, $k \in \mathbb{R}$, then

$$(ku) \cdot (1,-2,1) = k \cdot (u \cdot (1,-2,1)) = k \cdot 0 = 0$$

$\therefore ku \in T$. $\therefore T$ closed under scalar mult by s.

$\therefore T$ is a subspace of $\mathbb{R}^3$

<u>Remarks</u> 1) <u>Any</u> plane through origin is a subspace of $\mathbb{R}^3$ - there's nothing special about $(1,-2,1) = n$.

2) If a plane in $\mathbb{R}^3$ doesn't contain 0, it's <u>not</u> a subspace of $\mathbb{R}^3$

3) Any line in $\mathbb{R}^3$ (or $\mathbb{R}^2$) thru 0 is a s.s. of $\mathbb{R}^3$ (or $\mathbb{R}^2$)
(Same remark about lines not containing 0)

e.g. $V = F[0,1]$, $W = \{ f \in F[0,1] \mid f(0) = 0 \}$

0 : $0(0) = 0$ so $0 \in W$
$+$: $f, g \in W \Rightarrow (f+g)(0) = f(0) + g(0) = 0$ $\therefore f+g \in W$
k : $f \in W, k \in \mathbb{R} \Rightarrow (kf)(0) = k f(0) = k \cdot 0 = 0$ $\therefore kf \in W$

$\therefore W$ is a subspace of $F[0,1]$

Transpose (P.7) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t := \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; can do it for $m \times n$ matrices 7

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e.g. $S = \{ A \in M_{22} \mid A^t = A \}$ Symmetric 2×2 matrices

$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}.$

0: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \therefore 0 \in S$

$+$: $\begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ b' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ b+b' & d+d' \end{bmatrix} \in S \therefore S$ closed under $+$

$k\cdot$: $k \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{pmatrix} ka & kb \\ kb & kd \end{pmatrix} \in S \therefore$ S closed under mult. by scalar.

$\therefore S$ is a subspace of M_{22}

e.g. trace of square matrix   (can do it for $n \times n$'s)

$\mathrm{trace} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$;

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\mathrm{tr} A = a + d$ see #.

$SL_2 = \{ A \in M_{22} \mid \mathrm{tr}\, A = 0 \}$ is a subspace of M_{22}

$\left(= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \right)$ . . .

Spans & linear combinations (P. 299) Infinite $\to$ finite

Recall

$T = \{(x,y,z) \mid x-2y+z=0\}$ SS of $\mathbb{R}^3$

$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a,b,d \in \mathbb{R} \right\}$ SS of M_{22}

Rewrite T: $T = \{(x,y,z) \mid x = 2y - z\}$

$= \{(2y-z, y, z) \mid y, z \in \mathbb{R}\}$ no conditions left

$= \{(2y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{y(2,1,0) + z(-1,0,1) \mid y, z \in \mathbb{R}\}$

linear comb^n of $(2,1,0)$ & $(-1,0,1)$

"$= \text{span}\{(2,1,0), (-1,0,1)\}$"

$S = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a,b,d \in \mathbb{R} \right\}$

"$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$"

Def^{n}s (P. 299) (1) If $v_1, \ldots, v_m$ are vectors in a v.s. V, and $a_1, \ldots, a_m$ are scalars, the vector

$a_1 v_1 + \cdots + a_m v_m$ is called a

linear combination of $v_1, \ldots, v_m$.

(2) $\{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbb{R}\}$ is called the span of $v_1, \ldots, v_m$; we write

$\text{span}\{v_1, \ldots, v_m\} = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbb{R}\}$

$\{v_1, \ldots, v_m\}$ is called a spanning set for $\text{span}\{v_1, \ldots, v_m\}$

OR " spans "

(3) A vector space (or subspace) W is spanned by $v_1, \dots, v_m \in W$ if $W = \text{span}\{v_1, \dots, v_m\}$;
"$\{v_1, \dots, v_m\}$ spans W"

e.g. $T = \{y(2,1,0) + z(-1,0,1) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{\underset{v_1}{(2,1,0)}, \underset{v_2}{(-1,0,1)}\}$; $\{v_1, v_2\}$ spans T

$\{v_1, v_2\}$ is a spanning set for T

$S = \text{Span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{M_3}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

$\{M_1, M_2, M_3\}$ spans S; S is spanned by M_1, M_2 & M_3.

e.g. Let $L = \text{span}\{(0,1,0)\} = \{y(0,1,0) \mid a \in \mathbb{R}\}$
$= \{(0,y,0) \mid y \in \mathbb{R}\}$ (y-axis)

is a s.s. of $\mathbb{R}^3$! (like others...)

Thm 4 P. 299 Let V be a v.s. If $\{v_1, \dots, v_m\} \subset V$, then

(1) $U = \text{span}\{v_1, \dots, v_m\}$ is always a subspace of V

(2) If W is any s.s. of V s.t. $W \supseteq \{v_1, \dots, v_m\}$ then
$W \supseteq U$

i.e. $U = \text{span}\{v_1, \dots, v_m\}$ is the smallest s.s of V containing $v_1, \dots, v_m$.

Pf. (1) (Rem s.s. test) I $0 = 0v_1 + 0v_2 + \dots + 0v_m \in U$

II If $u = a_1v_1 + \dots + a_mv_m$ & $v = b_1v_1 + \dots + b_mv_m$, then
$u + v = (a_1+b_1)v_1 + \dots + (a_m+b_m)v_m \in U$

III If $u = a_1v_1 + \dots + a_mv_m$ & $k \in \mathbb{R}$,
$ku = (ka_1)v_1 + \dots + (ka_m)v_m \in U$.

(2) - exercise.

e.g. $\{(x,y,x-y) \mid x,y \in \mathbb{R}\} = \text{span}\{(1,0,1),(0,1,-1)\}$

$\therefore$ is a S.S. of $\mathbb{R}^3$

e.g. $V = F(\mathbb{R})$, $H = \text{span}\{f,g\}$ is a S.S. of $F(\mathbb{R})$!

$f(x) = \cos x$ $\quad = \{af + bg \mid a,b \in \mathbb{R}\}$

$g(x) = \sin x$

(ex. show, if $h(x) = \sin(x+1)$, $= (\sin x)(\cos 1) + (\sin 1)\cos x$

then $h \in H$!

hint: trig identity for $\sin(a+b)$)

$= a \sin x + b \cos x$

$(a = \cos 1,\ b = \sin 1)$

<u>note</u>: if $k(x) = 1$, $\forall x \in \mathbb{R}$, then $k \notin H$:

$k = af + bg$, fixed $a,b \in \mathbb{R}$

Suppose $k(x) = a\cos x + b \sin x$, $\forall x \in \mathbb{R}$

Choose convenient values of x:

$x = 0$: $k(0) = 1 = a \cdot \cos 0 + b \cdot \sin 0$

$\Rightarrow 1 = a$

$x = \frac{\pi}{2}$: $k(\frac{\pi}{2}) = 1 = a\cos \pi/2 + b \sin \pi/2$

$\Rightarrow 1 = b$

$x = \pi$: $k(\pi) = 1 = a \cos\pi + b\sin\pi$

$\Rightarrow 1 = -a$

C!

$\therefore k \notin \text{span}\{f,g\} = H$.

e.g.