CARLETON UNIVERSITY

FINAL EXAMINATION
December 2008

DURATION: 3 HOURS

Department Name and Course Number: Mathematics and Statistics, MATH 1005F
Course Instructor: Dr. A. Alaca
PART I: Multiple Choice Questions. No partial marks. Circle the correct answer.

[3] **1.** What is the orthogonal trajectories of the family of curves $y = ke^x$?

(a) $y^2 - x = c$ **(b)** $y^2 + 2x = c$ **(c)** $y + x^2 = c$ **(d)** $y + 2x^2 = c$ **(e)** $y^2 + x^2 = c$

[3] **2.** If y is the solution of the initial value problem $y' = 2xy$, $y(0) = 1$, what is $y(1)$?

(a) e **(b)** e^{-1} **(c)** $-e$ **(d)** e^2 **(e)** $2e$

[3] **3.** Let $y'' - 6y' + 8y = e^{4x}$. Which of the following is an appropriate form of a particular solution y_p when using the method of undetermined coefficients?

(a) $y_p = Ae^{4x}$

(b) $y_p = Axe^{4x}$

(c) $y_p = Ax^2e^{4x}$

(d) $y_p = (Ax + B)e^{4x}$

(e) $y_p = Ae^{4x} + Be^{-4x}$

[3] **4.** What is the general solution of $x^2y'' + 5xy' + 5y = 0$, $x > 0$?

(a) $y = x^{-1}\Big[c_1 \cos(\ln x) + c_2 \sin(\ln x)\Big]$

(b) $y = x^{-1}\Big[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\Big]$

(c) $y = x^{-2}\Big[c_1 \cos(\ln x) + c_2 \sin(\ln x)\Big]$

(d) $y = e^{-2x}\Big[c_1 \cos(x) + c_2 \sin(x)\Big]$

(e) $y = e^{-x}\Big[c_1 \cos(\ln x) + c_2 \sin(\ln x)\Big]$

[3] **5.** If y is the solution of the initial-value problem $y'' - 10y' + 25y = 0$, $y(0) = 1$, $y'(0) = 6$, then what is $y(1)$?

(a) e^5 **(b)** $2e^5$ **(c)** $5e^5$ **(d)** $e^5 + e^{-5}$ **(e)** e^{-5}

[3] **6.** What is the sum of the series $\sum_{n=0}^{\infty} \frac{4 \cdot 3^n}{5^{n+1}}$?

(a) 1 **(b)** 2 **(c)** $\frac{1}{2}$ **(d)** $\frac{6}{5}$ **(e)** $\frac{5}{6}$

[3] **7.** What is the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n\, 3^n}$?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3 (e) ∞

[3] **8.** You are given that the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}}$ is $R = 1$. What is the interval of convergence?

(a) $(-1, 1)$ **(b)** $[-1, 1)$ **(c)** $(-1, 1]$ **(d)** $(0, 2)$ **(e)** $(0, 2]$

[3] **9.** Which of the following is a power-series representation of the function $f(x) = \frac{x}{1 - x^4}$ about the point $a = 0$?

(a) $\sum_{n=0}^{\infty} x^{4n+1}$ **(b)** $\sum_{n=0}^{\infty} (-1)^n x^{4n+1}$ **(c)** $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ **(d)** $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ **(e)** $\sum_{n=0}^{\infty} \frac{x^{4n}}{n!}$

[3] **10.** What is the coefficient of x^3 in the Binomial series of the function $f(x) = \frac{1}{(1+x)^4}$ about the point $a = 0$?

(a) 120 **(b)** -120 **(c)** 20 **(d)** -20 **(e)** 10

PART II: Long answer questions. Show all your work.

[6] **1.** Solve the initial-value problem $y' - 2xy = 2x\, e^{x^2}$, $y(0) = 3$.

[6] **2.** Find the general solution of the equation $y' - \frac{1}{x} y = y^2$.

[6] **3.** Find the general solution of the equation $y' = \frac{xy + 3y^2}{x^2 + xy}$.

[10] **4(a)** Find the general solution of the equation $3x^2y^2 + 8xy^5 + (2x^3y + 20x^2y^4)y' = 0$.

(b) Show that the differential equation $3xy + 2y^2 + (x^2 + 2xy)y' = 0$ is not exact, and find an integrating factor which makes it exact. Write down the new exact differential equation, but do not solve it.

[10] **5(a)** Find two linearly independent solutions of the equation $y'' - 6y' + 9y = 0$.

(b) Find a particular solution of the equation $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^4},\ x > 0$ by using the method of variation of parameters.

(c) Write down the general solution of $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^4},\ x > 0$.

[12] **6.** For each one of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^4 + n^3 + n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{4^n}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

[6] **7.** Find the Taylor series of the function $f(x) = e^x$ about the point $a = 3$.

[6] **8.** Find the Taylor series of the function $f(x) = \ln(1 + x)$ about the point $a = 0$.

[8] **9.** Solve the initial-value problem $y'' + xy' + y = 0,\ y(0) = 0,\ y'(0) = 1$.

Answers to the multiple choice questions

1. (b)	6. (b)
2. (a)	7. (d)
3. (b)	8. (e)
4. (c)	9. (a)
5. (b)	10.(d)

Answers to the long answer questions

1. $y = e^{x^2}(x^2 + 3)$.

2. $y = \dfrac{x}{c - x^2/2}$.

3. $y = cx^3 e^{x/y}$.

4(a) $x^3y^2 + 4x^2y^5 = c$.

4(b) $I(x) = x$. New equation is $3x^2y + 2xy^2 + (x^3 + 2x^2y)y' = 0$.

5(a) $y_1 = e^{3x},\ y_2 = xe^{3x}$.

5(b) $y_p = \dfrac{e^{3x}}{6x^2}$.

5(c) $y = c_1e^{3x} + c_2xe^{3x} + \dfrac{e^{3x}}{6x^2}$.

6. (a) converges conditionally (b) converges absolutely (c) diverges.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$.

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$.

9. $\displaystyle y = \sum_{k=0}^{\infty} \frac{(-1)^k}{3 \cdot 5 \cdot 7 \cdots (2k+1)}x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!}x^{2k+1}$.