[12 points] **1.** Each part is worth 3 points. Put your answer in the box provided. You **do not** need to provide reasoning.

(a) Write down **all** of the equilibrium solution of the following differential equation:

$$y' = e^y\, y\,(y^2 - 4).$$

Answer: $0, \ 2, \ -2$

(b) Write down the general solution of the equation:

$$y'' - 6y' + 9y = 0.$$

$\lambda_1 = \lambda_2 = 3$

Answer: $C_1 e^{3t} + C_2 t e^{3t}$

(c) If $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation

$$y'' + Ay' + By = 0$$

where A and B are constants. Find A and B.

$3, -3$ are roots of characteristic equ.

$(\lambda - 3)(\lambda + 3) = 0$

$\Rightarrow \lambda^2 - 9 = 0$

Answer: $A = 0, \quad B = -9$

(d) Suppose that $y(t)$ is a solution to the initial value problem:

$$(t-2)y'' + (\ln t)\,y = 0, \quad y(1) = 1, \quad y'(1) = 2.$$

What is the largest interval on which $y(t)$ is guaranteed to exist?

$y'' + \dfrac{\ln t}{t-2}\, y = 0.$

Answer: $(0, 2)$

[9 points] **2.** Solve the following differential equation:

$$\underbrace{(xy^2 + 3x^2y)}_{M}dx + (\underbrace{x^3 + x^2y}_{N})dy = 0.$$

$M_y = 2xy + 3x^2$, $\quad N_x = 3x^2 + 2xy \quad \therefore$ exact.

$f_x = M = xy^2 + 3x^2y$

$f = \frac{1}{2}x^2y^2 + x^3y + h(y)$

$f_y = x^2y + x^3 + h'$

$\quad \|$

$N = x^3 + x^2y \quad \therefore h'(y) = 0 \quad \therefore h(y) = c$

$\therefore \quad f = \frac{1}{2}x^2y^2 + x^3y + C$

$\therefore$ gene. sol'n : $\quad \frac{1}{2}x^2y^2 + x^3y + C = 0. \quad (*)$

Note: $f(x,y) = \frac{1}{2}x^2y^2 + x^3y + C$

is not a solution!

The sol'n $y(x)$ is defined

Implicitly by $(*)$

Note: One can also try:

$$\frac{dy}{dx} = -\frac{xy^2 + 3x^2y}{x^3 + x^2y}$$

homogeneous eqn. Set $v = \frac{y}{x}$. ...

[5 points] **3.** Find an integrating factor of the following differential equation:

$$(x+y)dx + (x(\ln x) - 2xy)\,dy = 0.$$

Do not solve the equation after finding the integrating factor.

$$\left[\begin{array}{l} \tilde{M} = \mu\cdot(x+y), \qquad \tilde{N} = \mu\cdot(x\ln x - 2xy) \\ \tilde{M}_y = \mu_y(x+y) + \mu \\ \tilde{N}_x = \mu_x(x\ln x - 2xy) + \mu(\ln x + 1 - 2y) \end{array}\right]$$

Sol.

$$M_y = 1, \qquad N_x = \ln x + 1 - 2y$$

$$\frac{M_y - N_x}{N} = -\frac{\ln x - 2y}{x(\ln x - 2y)} = -\frac{1}{x}$$

$$\therefore \quad \mu' = -\frac{1}{x}\mu$$

$$\therefore \quad (\ln\mu)' = -(\ln x)' \qquad \therefore \quad \mu = \frac{1}{x}$$

[8 points] **4.** Solve the following differential equation:

$$(x \ln x)\, y' + y = 3x^3$$

for $x > 1$.

$$y' + \frac{1}{x \ln x} y = \frac{3x^2}{\ln x}$$

$$\mu' = \frac{1}{x \ln x} \mu$$

$$\therefore \ln|\mu| = \int \frac{1}{x \ln x}\, dx$$

$$= \ln|\ln x| + C$$

$$\therefore |\mu| = e^C |\ln x|, \qquad \text{take } \mu = \ln x$$

$$\ln x \cdot y' + \frac{1}{x} y = 3x^2$$

$$(\ln x \cdot y)' = 3x^2$$

$$\ln x \cdot y = x^3 + C$$

$$y = \frac{x^3 + C}{\ln x}, \qquad C: \text{arbitrary constant.}$$

[16 points] **5.** Consider the following differential equation, where $\alpha > 1$ is some fixed constant:

$$\frac{dy}{dt} = \frac{(y-1)(y-\alpha)}{1-\alpha}, \quad t \geq 0.$$

Show all your work.

(a) [2 points] Find all equilibrium solutions of the equation.

$y = 1, \quad y = \alpha.$

(b) [3 points] Find all ranges of y for which the solutions of this equation are strictly decreasing.

$$\frac{dy}{dt} < 0 \quad \Longrightarrow \quad \frac{(y-1)(y-\alpha)}{1-\alpha} < 0$$

$1 - \alpha < 0$ since $\alpha > 1$

$\therefore \quad (y-1)(y-\alpha) > 0$

$\therefore$

$y > \alpha$ or $y < 1$.

Note: no need to compute 2^{nd} derivative

(c) [8 points] Solve the initial value problem for $t \geq 0$:

$$\frac{dy}{dt} = \frac{(y-1)(y-\alpha)}{1-\alpha}, \quad y(0) = 2\alpha.$$

$$\frac{1-\alpha}{(y-1)(y-\alpha)} \, dy = dt.$$

Partial fraction:

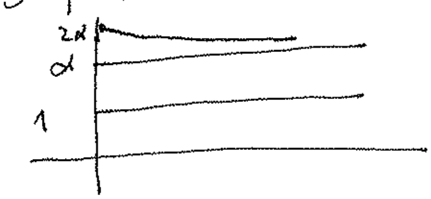
$$\frac{1-\alpha}{(y-1)(y-\alpha)} = \frac{A}{y-1} + \frac{B}{y-\alpha}$$

$$(1-\alpha) = A(y-\alpha) + B(y-1)$$

Take $y = 1 \Rightarrow A = 1.$

$y = \alpha \Rightarrow B = -1$

$$\therefore \int \frac{1-\alpha}{(y-1)(y-\alpha)} \, dy = \int \frac{dy}{y-1} - \int \frac{dy}{y-\alpha}$$



$$\therefore \ln \left| \frac{y-1}{y-\alpha} \right| = t + C$$

$$y(t) > \alpha \Rightarrow \frac{y-1}{y-\alpha} = e^{t+C}$$

Invoking initial condition $y(0) = 2\alpha \Rightarrow \frac{2\alpha - 1}{\alpha} = e^C$

$$\therefore y(t) = \frac{1-(2\alpha-1)e^t}{1 - \frac{2\alpha-1}{\alpha} e^t}$$

(d) [3 points] Find $\lim_{\alpha \to 1^+} y_\alpha(t)$, where $y_\alpha(t)$ is the solution to the initial value problem from part (c).

$$\lim_{\alpha \to 1^+} y_\alpha(t) = \lim_{\alpha \to 1^+} \frac{1-(2\alpha-1)e^t}{1 - \frac{2\alpha-1}{\alpha} e^t} = \frac{1-(2\cdot 1-1)e^t}{1 - \frac{2\cdot 1 - 1}{1} e^t}$$

$$= 1.$$