[12 points] **1.** Each part is worth 3 points. Put your answer in the box provided. You **do not** need to provide reasoning.

(a) Find the general solution of the following system of equations:

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{x}(t).$$

$\lambda_1 = 2$: $\begin{pmatrix} 2-2 & 1 \\ 0 & -1-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow y = 0.$ $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Answer: $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-t}$

$\lambda_2 = -1$: $\begin{pmatrix} 2+1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 3x + y = 0$

$\vec{V}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(b) Find the Laplace transform of the function f, where f is defined by

$$f(t) = \begin{cases} 0, & 0 \le t < 6 \\ (t-5)^2, & t \ge 6. \end{cases}$$

Answer: $e^{-6s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right), \; s > 0$

$$f(t) = (t-5)^2 u_6(t)$$
$$= [(t+1)-6]^2 u_6(t)$$
$$\mathcal{L}\{f(t)\} = e^{-6s}\mathcal{L}\{(t+1)^2\}$$
$$= e^{-6s}\mathcal{L}\{t^2+2t+1\}$$
$$= e^{-6s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right), \quad s > 0$$

or:

$$f(t) = [(t-6)+1]^2 u_6(t)$$
$$= u_6(t)(t-6)^2 + 2u_6(t)(t-6) + u_6(t)$$
$$\therefore \mathcal{L}\{f\} = e^{-6s}\cdot\frac{2}{s^3} + e^{-6s}\frac{2}{s^2} + e^{-6s}\frac{1}{s}$$

(c) Suppose that the vibration of a mass on a spring is modeled by the equation

$$y'' + y' + 2ky = 0.$$

Find all values of k for which this system is underdamped (this means: the mass oscillates about its equilibrium position).

$1 - 4 \cdot 2k < 0$

$\Rightarrow \quad k > \frac{1}{8}$

Answer: $k > \frac{1}{8}$

(d) For the system of equations given by

$$\mathbf{x}'(t) = \begin{pmatrix} -10 & 5 \\ 0 & -5 \end{pmatrix} \mathbf{x}(t)$$

state whether the equilibrium solution (0, 0) is an unstable node, an asymptotically stable node or a saddle point.

$\lambda_1 = -10, \quad \lambda_2 = -5$

distinct, negative

Answer: asy. stable node

[9 points] **2.** Use the method of **undetermined coefficients** to find a particular solution of

$$y'' - 2y' + y = 3e^t + e^{3t}.$$

Any solution that does not use the method of undetermined coefficients will receive a mark of zero. Show all your work.

homog. eqn: $y'' - 2y' + y = 0$

Char. eqn: $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda_1 = \lambda_2 = 1$$

a fund. set of sol'n : $e^t, \ te^t$

For non-homogeneous eqn:

$$y_p = At^2e^t + Be^{3t}$$

$$y_p' = At^2e^t + 2Ate^t + 3Be^{3t}$$

$$y_p'' = At^2e^t + 4Ate^t + 2Ae^t + 9Be^{3t}$$

$$\therefore \quad At^2e^t + 4Ate^t + 2Ae^t + 9Be^{3t} - 2(At^2e^t + 2Ate^t + 3Be^{3t}) + At^2e^t + Be^{3t} = 3e^t + e^{3t}$$

$$\therefore \begin{cases} 2A = 3 \\ 4B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{3}{2} \\ B = \frac{1}{4} \end{cases}$$

$$\therefore \quad y_p(t) = \frac{3}{2}t^2e^t + \frac{1}{4}e^{3t}$$

[12 points] **3.** Use the method of **variation of parameters** to find the general solution of the equation

$$y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{\ln t}{t^3}.$$

You may use the fact that $\{\frac{1}{t}, \frac{1}{t^2}\}$ is a fundamental set of solutions of the homogeneous equation $y'' + \frac{4}{t}y' + \frac{2}{t^2}y = 0$. Evaluate all integrals in your final answer. Show all your work.

$$y = \frac{1}{t}u_1 + \frac{1}{t^2}u_2$$

$$\begin{cases} \frac{1}{t}u_1' + \frac{1}{t^2}u_2' = 0 \\ -\frac{1}{t^2}u_1' - \frac{2}{t^3}u_2' = \frac{\ln t}{t^3} \end{cases} \Rightarrow \begin{cases} tu_1' + u_2' = 0 \\ tu_1' + 2u_2' = \ln t \end{cases}$$

$$\Rightarrow \begin{cases} u_1' = \frac{\ln t}{t} \\ u_2' = -\ln t \end{cases} \Rightarrow \quad u_1(t) = \int \frac{\ln t}{t}\,dt = \int \ln t \, d\ln t = \frac{1}{2}(\ln t)^2 + C_1$$

$$u_2(t) = -\int \ln t \, dt = -t\ln t + \int t \, d\ln t = -t\ln t + \int 1 \cdot dt = -t\ln t + t + C_2$$

$$\therefore \ y(t) = \frac{(\ln t)^2}{2t} + \frac{C_1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{C_2}{t^2}$$

C_1, C_2 : constants

[12 points] **4.** Solve the following initial value problem:

$$y'' + 4y = u_\pi(t) - u_{2\pi}(t), \quad y(0) = 1, \; y'(0) = 0$$

where $u_c(t)$ is the unit step function for $c = \pi$ and $c = 2\pi$.

$$s^2 \mathcal{L}\{y\} - s + 4\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\mathcal{L}\{y\} = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+4)} + \frac{s}{s^2+4}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad 1 = A(s^2+4) + s(Bs+C)$$

Take:
$s=0$: $A = \frac{1}{4}$
$s=1$: $1 = 5/4 + B + C$
$s=-1$: $1 = 5/4 + B - C$
$\Rightarrow B = -\frac{1}{4}$, $C = 0$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\left(e^{-\pi s} - e^{-2\pi s}\right)\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right)\right\} + \cos 2t$$

$$= \frac{1}{4}u_\pi(t) - \frac{1}{4}u_{2\pi}(t) - \frac{1}{4}u_\pi(t)\cos 2(t-\pi) + \frac{1}{4}u_{2\pi}(t)\cos 2(t-2\pi) + \cos 2t$$

$$= \frac{1}{4}u_\pi(t) - \frac{1}{4}u_{2\pi}(t) - \frac{1}{4}u_\pi(t)\cos 2t + \frac{1}{4}u_{2\pi}(t)\cos 2t + \cos 2t$$

$$= \frac{1}{4}u_\pi(t)(1-\cos 2t) - \frac{1}{4}u_{2\pi}(t)(1-\cos 2t) + \cos 2t$$

$$= \frac{1}{4}\left(u_\pi(t) - u_{2\pi}(t)\right)(1-\cos 2t) + \cos 2t$$

[5 points] **5.** Let f be a continuous function and let $\int_0^\infty f(x)dx$ be convegent. Define

$$g(t) = \int_0^{at} f(x)dx$$

for some positive constant a. Suppose that the Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for $s > 0$. Use the definition of Laplace transforms to show that

$$\mathcal{L}\{g(t)\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(\frac{s}{a})$$

for $s > 0$. (Hint: you may need to use integration by parts)

$$\mathcal{L}\{g(t)\}(s) = \int_0^\infty e^{-st} \int_0^{at} f(x)\,dx\,dt$$

$$= -\frac{1}{s}\int_0^\infty \int_0^{at} f(x)\,dx\,d\,e^{-st}$$

$$\overset{\text{Integration by parts}}{=} -\frac{1}{s}\left(e^{-st}\int_0^{at} f(x)\,dx \Big|_{t=0}^{t=\infty} - \int_0^\infty e^{-st} \frac{d}{dt}\int_0^{at} f(x)dx\,dt \right)$$

fundamental Theorem of Calculus

$$\underset{\int_0^\infty f(x)dx < \infty}{\overset{a>0}{=}} -\frac{1}{s}\left(0 - 0 - \int_0^\infty e^{-st}\, a\cdot f(at)\,dt\right)$$

$$\overset{y=at}{=} \frac{1}{s}\int_0^\infty e^{-\frac{s}{a}y} f(y)\,dy$$

$$= \frac{1}{s}\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right)$$