MATH1005C — Solution-Test 2 — 4:35–5:25, Feb. 27 2013

Total: 20 marks

Question 1. [3] The general solution of $y'' + 4y' + 5y = 0,\ x > 0$ is given by

A. $c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin x$ (**) B. $c_1 e^{-2x} + c_2 e^{x}$

C. $c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x)$ D. $c_1\cos(2x) + c_2\sin(2x)$

E. None of these

Question 2. [3] The general solution of $x^2 y'' - 7xy' + 16y = 0,\ x > 0$ is given by

A. $c_1 e^{4x} + c_2 x e^{4x}$ B. $c_1 \ln(x^4) + c_2 x \ln(x^4)$

C. $c_1 x^4 + c_2 x^5$ D. $c_1 x^4 + c_2 x^4 \ln x$ (**)

E. None of these

Question 3. [4] Find the general solution of the equation

$$y'' + y' - 2y = 9e^x$$

Sol:
To find y_h, consider the homogeneous equation: $y'' + y' - 2y = 0$, the indicial equation is $r^2 + r - 2 = (r-1)(r+2) = 0$ then there are two roots $r = 1,\ r = -2$, and general solution is

$$y_h = c_1 e^x + c_2 e^{-2x}.$$

Particular solution is in the form $y_p = Axe^x,\ y' = Ae^x + Axe^x,\ y'' = Ae^x + Ae^x + Axe^x$, after plug in $A = 3$. The general solution is

$$y = c_1 e^x + c_2 e^{-2x} + 3xe^x.$$

Question 4. [5] Find the general solution by variation of parameters of the equation

$$x^2 y'' - 4xy' + 6y = x^4 e^x$$

Sol:
For homogeneous equation $x^2y'' - 4xy' + 6y = 0, y = x^r$ is the solution and indicial equation is $r(r-1) - 4r + 6 = r^2 - 5r + 6 = (r-2)(r-3) = 0$, so $y = x^2$ and $y = x^3$ are two linearly

independent solutions and general solution is $y_h = c_1x^2 + c_2x^3$.
To find a particular solution; the standard form is $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2e^x$ so $f(x) = x^2e^x$ and $W(x) = y_1y_2' - y_1'y_2 = x^2(3x^2) - x^3(2x) = x^4$.
Then $u_1 = \int \frac{-y_2f(x)}{W(x)}dx = -\int xe^x dx = e^x - xe^x$ (integration by part) and $u_2 = \int \frac{y_1f(x)}{W(x)}dx = \int e^x dx = e^x$, so $y_p = (e^x - xe^x)x^2 + x^3e^x = x^2e^x$. The general solution is

$$y = x^2e^x + c_1x^2 + c_2x^3$$

Question 5. [5] Solve the initial-value problem

$$(4xy)dx + (6x^2 + y)dy = 0, \quad y(1) = 1$$

Sol:
$(4xy)dx + (6x^2 + y)dy = 4xy + (6x^2 + y)y' = 0$ where $P(x, y) = 4xy, Q(x, y) = 6x^2 + y$ then $P_y = 4x,\ Q_x = 12x$ which are not equal and equation is not exact.
$\frac{Q_x - P_y}{P} = \frac{8x}{4xy} = \frac{2}{y}$ is independent of x, so integration factor exists.
$\frac{I'(y)}{I(y)} = \frac{2}{y} \to \ln(I(y)) = 2\ln y \to I(y) = y^2$, the equation becomes

$$4xy^3 + (6x^2y^2 + y^3)y' = 0$$

which is exact. Then

$$\begin{aligned} f_x &= 4xy^3 \to f(x, y) = 2x^2y^3 + g(y) \\ f_y &= 6x^2y^2 + g'(y) = 6x^2y^2 + y^3 \to g'(y) = y^3 \to g(y) = \frac{1}{4}y^4 + c \\ f(x, y) &= 2x^2y^3 + \frac{1}{4}y^4 + c \to 2x^2y^3 + \frac{1}{4}y^4 = k \text{ or } 8x^2y^3 + y^4 = k \\ y(1) &= 1 \to k = 9 \end{aligned}$$