

MATH1005C — Solution-Test 1 — 4:35–5:25, Jan. 30 2013

Total: 20 marks

Question 1. Consider the equation $\frac{y^2}{x^2} - y' = 0$.

- (a) Solve it as separable equation.
- (b) Solve it as homogeneous equation.
- (c) Solve it as a Bernoulli equation.
- (d) Find the orthogonal trajectories of the one-parameter family of curves defined by the general solution.

Solution:

(a) $\frac{1}{y^2}y' = \frac{1}{x^2} \rightarrow \int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx \rightarrow \frac{-1}{y} = \frac{-1}{x} + c \rightarrow y = \frac{x}{1-cx}$.

(b) $y' = \left(\frac{y}{x}\right)^2 = u^2$ if $u = \frac{y}{x} \rightarrow y = ux \rightarrow y' = u + u'x = u^2 \rightarrow xu' = u^2 - u \rightarrow \frac{u'}{u(u-1)} = \frac{1}{x}$
 $\rightarrow \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du = \int \frac{1}{x} dx \rightarrow \ln|u-1| - \ln|u| = \ln|x| + c \rightarrow \ln\left(\frac{|u-1|}{|u|}\right) = \ln|x| + c$
 $\rightarrow \frac{|u-1|}{|u|} = c|x| \rightarrow u = \frac{1}{1-cx} \rightarrow y = \frac{x}{1-cx}$

(c) $y' = \frac{1}{x^2}y^2 \rightarrow \alpha = 2$ and $u = y^{-1} \rightarrow y = u^{-1} \rightarrow y' = -u^{-2}u' \rightarrow -u^{-2}u' = u^{-2}x^{-2}$
 $\rightarrow u' = -x^{-2} \rightarrow u = \frac{1}{x} + c \rightarrow 1/y = \frac{1}{x} + c \rightarrow y = \frac{x}{1+cx}$

(d) $y' = \frac{y^2}{x^2}$ so orthogonal trajectories are given by $y' = -\frac{x^2}{y^2}$ then $y^2y' = -x^2$
 $\rightarrow \int y^2 dy = \int -x^2 dx \rightarrow \frac{y^3}{3} = -\frac{x^3}{3} + c \rightarrow x^3 + y^3 = c$

Question 2. Let $f(x, y) = \sin(xy) + \sqrt{xy}$, $x(t) = e^t + 1$ and $y(t) = \ln(t)$. Determine $\frac{d}{dt}f(x(t), y(t))$.

Solution:

$$\frac{d}{dt}f(x(t), y(t)) = (y \cos(xy) + \frac{\sqrt{y}}{2\sqrt{x}})(e^t) + (x \cos(xy) + \frac{\sqrt{x}}{2\sqrt{y}})\left(\frac{1}{t}\right)$$

Question 3. Solve the initial-value problem

$$\cos(x)y' + \sin(x)y = \cos^4(x), \quad y(\pi) = \frac{\pi}{2}$$

Solution:

$\cos(x)y' + \sin(x)y = \cos^4(x) \rightarrow y' + \frac{\sin(x)}{\cos(x)} = \cos^3(x)$, then

$$I(x) = e^{\int p(x)dx} = e^{\int \frac{\sin(x)}{\cos(x)} dx} = e^{\ln |\sec(x)|} \rightarrow I(x) = \sec(x)$$
$$y = \frac{1}{I(x)} \int Q(x)I(x)dx = \cos(x) \int \cos^2(x)dx = \cos(x) \int \frac{1+\cos(2x)}{2} dx = \cos(x) \left[\frac{1}{2}x + \frac{1}{4} \sin(2x) + c \right]$$
$$y(\pi) = \frac{\pi}{2} \rightarrow \pi/2 = -1(\pi/2 + c) \rightarrow c = -\pi$$