COMP 1805 Discrete Structures I

Assignment 4

Date Due: March 26, 2013

Time Due: Before 2pm in the Boxes in Herzberg Room 3115

Write down your name and student number on **every** page. The questions should be answered in order and your assignment sheets must be stapled.

1. Show the following:
   (a) $\sum_{i=1}^{n}(3i+2n)$ is $O(n^2)$.
   (b) $\sum_{k=1}^{n}(k/3)$ is $\Omega(n^2)$
   (c) $\sum_{j=1}^{n}\sum_{k=1}^{n} 7n$ is $\Theta(n^3)$

2. Suppose that an algorithm uses $2n^2+3^n$ bit operations to solve a problem of size n. Suppose that your machine can perform one bit operation in 10^{-9} seconds, how long does it take your algorithm to solve a problem of size given below. Note, if your algorithm takes more than 60 seconds, answer in minutes. For more than 60 minutes, answer in hours. For more than 24 hours, answer in days. For more than 365 days, answer in years. For more than 100 years, answer in centuries!
   (a) 10
   (b) 20
   (c) 50
   (d) 100
   (e) 1000

3. Determine whether or not the following are true. If true, then provide a derivation showing why. If false, then provide a derivation showing why. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.
   (a) 42 is $\Theta(1)$.
   (b) $(n-7)^2$ is $\Theta(n^2)$
   (c) $2\log(3n^3-n^2+43)$ is $\Theta(\log n)$.
   (d) $3n^2-n+5$ is $O(n^2)$.
   (e) $n^3/58-7n^2\log n$ is $\Theta(n^2)$.
   (f) $2n^4-5n^3\log n-3n^2+9n\log n$ is $\Omega(n^4)$
   (g) $6n^2-3n-4$ is $O(n^3)$.
   (h) n^2-13n is $\Omega(1)$.
   (i) $2n^{5/2}-n^2$ is $O(n^3\log n)$.
   (j) $n/6-27$ is $\Omega(n)$.

4. If $f(n)$ is $\Omega(n^2)$ and $g(n)$ is $O(n)$, determine whether or not the following are true. If you believe it is true, then provide a proof. If you think it is not true, than provide a counter-example.
   (a) $f(n)$ is $O(n)$.
   (b) $f(n)$ is $\Omega(1)$.

(c) $f(n)$ is $O(n^2)$.
(d) $g(n)$ is $\Omega(n)$.
(e) $g(n)$ is $\Omega(n^3)$.
(f) $g(n)$ is $O(n^3)$.

5. Prove by induction that $\sum_{i=1}^{n} 3/4^i < 1$ for all $n \geq 2$

6. Given a graph G with n vertices, where n is even, prove that if every vertex has degree $n/2+1$, then G must contain a 3-cycle. A 3-cycle is a set of 3 vertices, a, b, c such that ab is an edge, bc is an edge and ac is an edge.

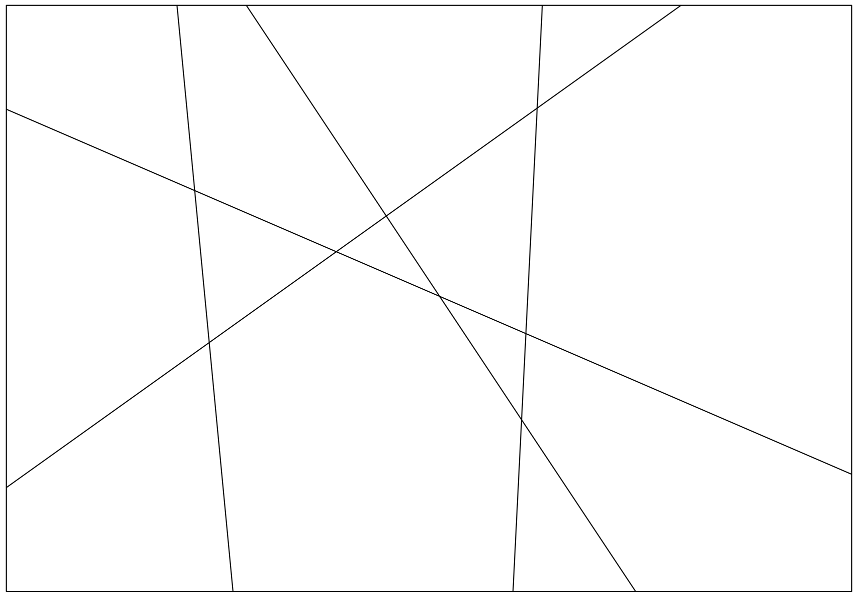
7. Given a graph $G = (V, E)$, the complement of G is the graph G' defined on the same vertex set V, however, an edge is present in G' provided that it is not in G. Prove that if G is not connected, then G' must be connected.

8. Prove that a graph $G = (V, E)$ that has no cycles and contains $|V| - 1$ edges must be a tree.

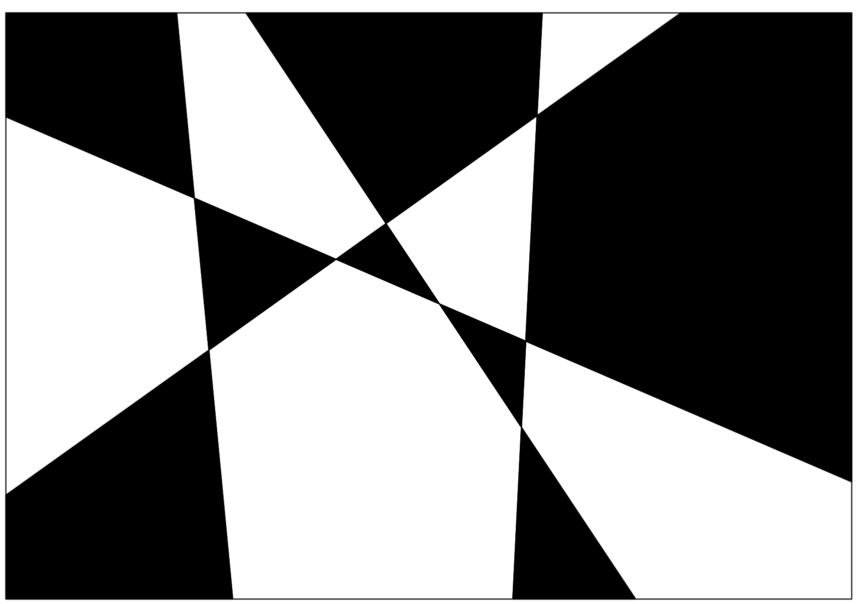
9. Consider the following recursive function:
Base Case: $F(0) = 0, F(1) = 1$.
Recursive Step: $F(n) = F(n-1) + F(n-2)$ for all $n \geq 2$.

(a) Prove by induction that $F(n) \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$
(b) Prove by induction that $F(n) \geq (1.5)^{n-1}$

10. Bonus Challenge Question (optional): Consider a rectangle R. Start drawing n line segments $l_1, l_2, \ldots, l_n$ such that each line segment has both endpoints on the boundary of the rectangle. Notice that this partitions the rectangle R into a bunch of regions. Suppose you have two colors, black and white. Prove by induction that you can always find a way to paint each of the regions with either black or white such that every two regions that share a piece of segment (not a point) gets a different color. In other words, two regions that share a piece of segment never get colored with the same color.



Rectangle cut by line segments. Each line segment has both endpoints on the boundary of the rectangle. The rectangle is decomposed into a bunch of regions.



A valid 2-coloring of the regions. A 2-coloring is valid provided that two regions that share an edge get different colors.