Solutions to the final examination for MATH 1009* EG, Fall 2012.

PART A. MULTIPLE CHOICE QUESTIONS. Each answer is worth 3.5 marks.

Answers: dcada bcbdb adacb daabc

1. Evaluate $\lim_{x\to\infty} \frac{3x^5 - x + 5}{1 + 3x^2 - x^4}$.

(a) -3. (b) 1. (c) 0. (d) ∞. (e) None of these.

Answer: (d)

2. The domain of the function $f(x) = \frac{3x}{\sqrt{(x-4)}}$ is

(a) $(0, 4)$, or equivalently, $\{x : 0 < x < 4\}$.

(b) $(-\infty, 4)$, or equivalently, $\{x : x < 4\}$.

(c) $(4, \infty)$, or equivalently, $\{x : x > 4\}$.

(d) $[4, \infty)$, or equivalently, $\{x : x \geq 4\}$.

(e) None of these.

Answer: (c)

3. Evaluate $\lim_{x\to 3} \frac{3 - x}{x^2 - 9}$.

(a) $-\frac{1}{6}$ (b) $\frac{1}{6}$ (c) 0 (d) ∞ (e) None of the above

Answer: (a)

4. The expression $\frac{x^{1.6} \cdot (x^2)^{-0.3}}{x^{-2}}$ simplifies to

(a) x. (b) $x^{1.9}$. (c) $x^{-1.92}$. (d) x^3. (e) None of these.

Answer: (d)

5. Which of the following is equal to $\log_5 \frac{1}{25}$?

(a) -2. (b) 2. (c) $-\frac{1}{2}$. (d) $\frac{1}{2}$. (e) None of these.

Answer: (a)

6. What is the slope of the curve $y = 3^{2x}$ at $x = 0$?

(a) 0. (b) $2 \ln 3$. (c) $\ln 3$. (d) 1. (d) Not defined.
(e) None of these.

Answer: (b)

7. Given that the profit function for a company is

$$P(x) = x^2 + \frac{100}{x} + 2x - 10,$$

find the **marginal profit** at the production level of $x = 10$ units.

(a) 120. (b) 210. (c) 21. (d) 23. (e) None of these

Answer: (c)

8. The graph of the function $y = \dfrac{3x - 1}{x - 2}$ has

(a) no horizontal asymptote and a vertical asymptote $x = 2$.

(b) a horizontal asymptote $y = 3$ and a vertical asymptote $x = 2$.

(c) a horizontal asymptote $y = 0$ and no vertical asymptote.

(d) neither horizontal no vertical asymptote.

(e) none of the these.

Answer: (b)

9. What are the critical numbers of the function $f(x) = \dfrac{7}{x - 4}$?

(a) $x = -7$. (b) $x = -4$. (c) $x = 4$. (d) No critical numbers. (e) None of these.

Answer: (d)

10. Let $f(x) = (3 + \log_3 x)^7$. Evaluate $f'(x)$, that is, find the derivative of f.

(a) $\dfrac{7(3 + \log_3 x)^6}{x}$ (b) $\dfrac{7(3 + \log_3 x)^6}{x \ln 3}$ (c) $\dfrac{(3 + \log_3 x)^6}{x \ln 3}$

(d) $7(3 + \log_3 x)^6 \ln 3$ (e) None of the above

Answer: (b)

11. Determine the interval(s) where the function $f(x) = x^4 - 2x^2 + 1$ is INCREASING.

(a) $(-1, 0) \bigcup (1, \infty)$, or equivalently, $\{x : \ -1 < x < 0, \ x > 1\}$.

(b) $(-\infty, -1) \bigcup (0, \infty)$, or equivalently, $\{x : \ x < -1, \ x > 0\}$.

(c) $(-1, 1)$, or equivalently, $\{x : -1 < x < 1\}$.

(d) $(0, 1)$, or equivalently, $\{x : \ x > 1\}$.

(e) None of these.

Answer: (a)

12. How many years would it take for the investment of \$2,000 to grow to \$6,000, if the interest is compounded semi-annually and the nominal interest rate is 4%? $A(t) = P\left(1 + \frac{r}{m}\right)^{mt}$.

(a) 11.8. (b) 17.6. (c) 21.9. (d) 27.7. (e) None of these.

Answer: (d)

13. Consider $f(x) = x^3 - 12x + 1$, where $x \in [-3, 3]$. This function has

(a) a global (absolute) minimum at $x = 2$ and a global (absolute) maximum at $x = -2$.

(b) a global (absolute) minimum at $x = -2$ and a global (absolute) maximum at $x = 2$.

(c) a global (absolute) minimum at $x = 3$ and a global (absolute) maximum at $x = -3$.

(d) a global (absolute) minimum at $x = -3$ and a global (absolute) maximum at $x = 3$.

(e) None of the above

Answer: (a)

14. The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of 1/2. The half-life of aspirin in a human body is 5 hours. Find its decay constant.

(a) 0.5. (b) 3.2. (c) 5.7. (d) 9.8. (e) None of these.

Answer: (c)

15. Find the infection point(s) of the function $f(x) = \frac{1}{6}x^4 - 9x^2 + 81.5$.

(a) $(3, 5)$ and $(-3, 5)$. (b) $(3, 14)$ and $(-3, 14)$. (c) $(9, 5)$.

(d) $(9, 14)$. (e) None of these.

Answer: (b)

16. Let $f(x,y) = x^3 + y^2 + 3x^2y$. Find $\dfrac{\partial f}{\partial x}$, or f_x.

(a) $2y + 3x^2$. (b) $3x^2 + 2y + 6xy$. (c) $3x^2 + 3y$.

(d) $3x^2 + 6xy$. (e) None of these.

Answer: (d)

17. Let $f(x,y) = e^{xy}$. What is $f_{yy}(2,1)$?

(a) $4e^2$. (b) $2e^2$. (c) e^2. (d) e. (e) None of these.

Answer: (a)

18. Consider the Cobb-Douglas production function $f(x,y) = 3x^{2/3}y^{1/3}$, where x is the number of units of labour and y is the number of units of capital. What is the marginal productivity of **labour** at $x = 8,\ y = 27$?

(a) 3. (b) 36. c) 64. (d) -18. (e) None of these.

Answer: (a)

19. The value of $\displaystyle\int_{-1}^{2} (2x+3)\,dx$ is

(a) 6. (b) 12. (c) -4. (d) 0. (e) None of these.

Answer: (b)

20. Find $\displaystyle\frac{d}{dx}\int_{2}^{x} \sqrt{t^2+t+1}\,dt$.

(a) $\dfrac{t}{\sqrt{t^2+t+1}}$. (b) $\dfrac{x}{\sqrt{x^2+x+1}} - \dfrac{2}{\sqrt{7}}$. (c) $\sqrt{x^2+x+1}$.

(d) $\sqrt{x^2+x+1} - \sqrt{7}$. (e) None of these.

Answer: (c)

PART B. For full marks, be sure to show all of your steps.

21. [**10 marks**] Use the method of Lagrange multipliers to find the maximum and the minimum values of the function

$$f(x, y) = (x-1)^2 + (y-2)^2$$

subject to the constraint

$$x^2 + y^2 = 45.$$

Solution:

$$F(x, y, \lambda) = (x-1)^2 + (y-2)^2 + \lambda(x^2 + y^2 - 45).$$

$$\left\{ \begin{array}{llcrl} F_x & = 2(x-1) & + & 2\text{x}\lambda = 0, & (1) \\ F_y & = 2(y-2) & + & 2\text{y}\lambda = 0, & (2) \\ F_\lambda & = x^2 & + & y^2 - 45 = 0. & (3) \end{array} \right.$$

From (1) $\Rightarrow \ \lambda = -\dfrac{2(x-1)}{2x}$, from (2) $\Rightarrow \ \lambda = -\dfrac{2(y-2)}{2y}$.

Equate λ to obtain $\dfrac{x-1}{x} = \dfrac{y-2}{y}$, or $y = 2x$.

Substituting $y = 2x$ into (3) yields

$$5x^2 = 45 \ \Rightarrow \ x^2 = 9 \ \Rightarrow \ x = \pm 3$$

.

Thus, when $x = 3$ then $y = 2x = 6$, and when $x = -3$ then $y = -6$.

The critical points are $(3, 6)$, $(-3, 6)$. Evaluate $f(x, y)$ at each of the critical points to find the minimum and the maximum value:

$f(3, 6) = 20$ is the minimum value; $f(-3, -6) = 80$ is the maximum value.

22. [10 Marks]

[4] **(a)** Find all the critical points of the function $f(x, y) = x^2 - 4xy + 2y^2 + 4x + 8y - 1$. (Do NOT classify them).

Solution:

$f_x = 2x - 4y + 4 = 0;$

$f_y = -4x + 4y + 8 = 0;$

Add the equations to get $-2x + 12 = 0$ or $x = 6$. Then from the first equation

$$4y = 2x + 4, \quad y = \frac{x}{2} + 1 = \frac{6}{2} + 1 = 4$$

.

Thus, the critical point is $(6, 4)$.

[6] **(b)** The function $f(x, y) = 2x^3 - 2xy + y^2 + 7$ has two critical points $(0, 0)$ and $(\frac{1}{3}, \frac{1}{3})$. Use the Second Derivative Test to classify the nature of each point, if possible.

The Second Derivative Test:

$f_x = 6x^2 - 2y;$

$f_y = -2x + 2y;$

$f_{xx} = 12x, \quad f_{yy} = 2, \quad f_{xy} = -2,$ so $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24x - 4.$

$D(0, 0) = -4 < 0,$ therefore $(0, 0)$ is a saddle point (neither a relative min nor a relative max).

$D\left(\frac{1}{3}, \frac{1}{3}\right) = 24 \cdot \left(\frac{1}{3}\right) - 4 = 4 > 0, \quad f_{xx} = 4 > 0,$ therefore this is a local (relative) minimum point.

23. [**4 marks**] Consider the equation $x^2 + 2xy^2 + y = 0$, where $y = y(x)$ is defined implicitly as a function of x. Find $\dfrac{dy}{dx}$.

Solution:

Rewrite the equation, stating that $y = y(x)$ is a function of x:

$x^2 + 2x \cdot [y(x)]^2 + y(x) = 0.$

Differentiate both parts of the above expression with respect to x:

$2x + 2[y(x)]^2 \cdot 1 + 2x \cdot 2y(x) \cdot y'(x) + y'(x) = 0.$

Solve for $y'(x)$:

$4xyy' + y' = -2x - 2y^2, \qquad y'(x) = \dfrac{-2x - 2y^2}{4xy + 1}.$

24. [**6 Marks**] Evaluate the following integrals:

[3] (a) $\displaystyle\int \frac{4x^4 - x + 5}{x}\,dx = 4\int x^3\,dx - \int x\,dx + 5\int \frac{1}{x}\,dx = 4 \cdot \frac{x^4}{4} - x + 5\ln|x| + C =$

$= x^4 - x + 5\ln|x| + C.$

[3] (b) $\displaystyle\int 6x^2(x^3 - 4)^7\,dx$

Let $u = x^3 - 4$, then $\dfrac{du}{dx} = 3x^2,\ du = 3x^2\,dx,\ dx = \dfrac{du}{3x^2}.$

$$\int 6x^2(x^3 - 4)^7\,dx = \int 6x^2(u)^7\,\frac{du}{3x^2} = 2\int u^7\,du = 2 \cdot \frac{u^8}{8} + C = \frac{(x^3 - 4)^8}{4} + C$$