Université d'Ottawa
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Canada's university

University of Ottawa
Faculty of Engineering

Department of
Civil Engineering

CVG 2181
Numerical Methods in Civil Engineering

FINAL EXAMINATION

Length of Examination: 3 hours **April 11, 2006, 14:00**

Professor: Darek Baingo

Family Name: ____

Other Names: ____

Student Number: ___________

Signature ___________________________

CLOSED BOOK.

If you do not understand a question, clearly state an assumption and proceed. Attempt all problems.

Use **5 decimal precision** for all calculations!

Non-programmable calculators are permitted. No other electronic devices are allowed.

Marks are as shown for each question. A total of 105/100 points is possible.

At the end of the exam, when time is up:

- Stop working and turn your exam upside down.
- Remain silent.
- Do not move or speak until all exams have been picked up, and a TA or the Professor gives the go-ahead to leave.

1. a) (10 pts.) Solve the following differential equation at $x = 0.6$ with a 3rd order Adams technique:

$$y' = x^3 + y^2$$

where $y(0) = 0$, $y(0.2) = 0.0004$, $y(0.4) = 0.0064$

b) (8 pts.) Solve the same equation with the 4th order Runge-Kutta technique. Perform one iteration with the initial point at $x = 0.4$ and a step size of $h = 0.2$

c) (4 pts.) What are the advantage(s) of multi-step methods over one-step methods? What are the advantage(s) of one-step methods over multi-step methods?

2. (15 pts.) Use the Romberg Integration method to estimate the integral below so that its error is of order of $O(h^8)$.

$$\int_0^{\pi} \sin x \, dx$$

3. (8 pts.) For the following initial value problem:

$$y'' = -2 \cdot y' + 4y - 8; \qquad y(0) = 5, \quad y'(0) = 0$$

Solve for y over the range $x = 0$ to $x = 2$ using Euler's method with a step size $h = 1$.

4. (10 pts.) The expression below has a root in the range of 0 to 2.

$$f(x) = x^2 e^{(x/2)} - 1$$

Conduct three iterations of the bisection method and indicate the points to be used in the next (fourth) iteration. What rate of convergence do you expect for this method (do not calculate).

5. (10 pts.) Consider the following:

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 9 & 4 & 2 \\ 8 & 5 & 1 \end{bmatrix} \; ; \; A^{-1} = \begin{bmatrix} -0.0606 & 0.3333 & -0.2424 \\ 0.0707 & -0.5556 & 0.6161 \\ 0.1313 & 0.1111 & -0.1414 \end{bmatrix} \; ; \; X^T = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix}$$

a) Calculate: $\|A\|_e$, $\|A\|_1$, $\|A\|_\infty$, and $\|X\|_\infty$

b) Calculate the condition number of A based on the $\| \cdot \|_1$ norms. What are the implications of this particular value of the condition number?

6. (20 pts.) For the given data set:

x	−1	0	1	2	3
y	0	4	6	6	4

a) (4 pts.) Integrate the data using an appropriate Simpson's Rule
b) (6 pts.) Derive a 2^{nd} order (quadratic) polynomial for the above points using Lagrange interpolation.
c) (6 pts.) Apply 3-point Gauss-Legendre quadrature to integrate the function obtained in b) over the range [-1, 3].
d) (4 pts.) Integrate the polynomial in b) analytically, and compare the value with results in a) and c). Discuss the observations.

7. a) (3 pts.) Estimate the first derivative of $f(x) = \ln(x)$ at $x = 3$ with an error of magnitude $O(h^4)$ using the central difference scheme and an interval of $h = 1$. Compare to the exact solution (error).

b) (10 pts.) Use the following experimental data to find the velocity <u>and</u> acceleration at $t = 10$ seconds:

Time t, (s)	0	2	4	6	8	10	12	14	16
Position x (m)	0	0.7	1.8	3.4	5.1	6.3	7.3	8.0	8.4

Use second-order correct (i) central finite-difference, and (ii) forward finite-difference.

c) (2 pts.) Explain why or why not numerical integration is more accurate than numerical differentiation.

8. (5 pts.) <u>**Bonus:**</u> Solve only <u>**one**</u> of the following problems, (a) <u>**or**</u> (b):

a) The concentration of pollutant bacteria c in a lake decreases according to:

$$c = 75e^{-1.5t} + 20e^{-0.075t}$$

Determine the time required for the bacteria concentration to be reduced to 15 using <u>two</u> iterations of the Newton-Raphson method, with an initial guess of $t = 6$

b) A civil engineer involved in a construction project requires 4800, 5800, and 5700 m^3 of sand, fine gravel, and coarse gravel, respectively. There are three pits from which these materials can be obtained. The composition of these pits is:

	Sand (%)	Fine Gravel (%)	Coarse Gravel (%)
Pit 1	55	30	15
Pit 2	25	45	30
Pit 3	25	20	55

How many cubic metres must be hauled from each pit to meet the engineer's needs? Set up the appropriate system of linear equations and solve using the two iterations of Gauss-Seidel method.

CVG 2181 - USEFUL EQUATIONS

$$x_r = \frac{x_l + x_u}{2}$$

$$x_{i+1} = g(x_i)$$

$$x_r = x_u - \frac{f(x_u)}{\left[f(x_l) - f(x_u)\right]}(x_l - x_u)$$

$$x_{i+1} = x_i - \frac{f(x_i)\left(x_{i-1} - x_i\right)}{f(x_{i-1}) - f(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Delta x = x_{n+1} - x_n = \frac{-f(x_n)\cdot f'(x_n)}{\left[f'(x_n)\cdot f'(x_n) - f''(x_n)\cdot f(x_n)\right]}$$

$$\Delta x = x_{n+1} - x_n = \frac{-2f(x_n)\cdot f'(x_n)}{\left[2f'(x_n)\cdot f'(x_n) - f''(x_n)\cdot f(x_n)\right]}$$

$$l_{i1} = a_{i1} \;;\quad u_{1j} = \frac{a_{1j}}{l_{11}} \quad j \geq 2 \;;\quad l_{ij} = a_{ij} - \sum_{k=1}^{j-1}(l_{ik}\cdot u_{kj}) \quad j \leq i \;;\quad u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1}(l_{ik}\cdot u_{kj})}{l_{ii}} \quad i \leq j$$

$$x_i^{(n+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{a_{ij}}{a_{ii}}x_j^{(n)}$$

$$\left|a_{ii}\right| > \sum_{\substack{j=1\\ j\neq i}}^{n}\left|a_{ij}\right| \quad \text{for } i = 1,\ 2,\ 3,\ldots,n$$

$$x_i^{(n+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}x_j^{(n+1)} - \sum_{j=1+1}^{n}\frac{a_{ij}}{a_{ii}}x_j^{(n)}$$

$$\frac{\max}{\forall \mathrm{i}}\left(\frac{1}{\left|a_{ii}\right|}\sum_{\substack{j=1\\ i\neq j}}^{n}\left|a_{ij}\right|\right) < 1$$

$$f_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

$$f_n(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + \ldots$$
$$+ f[x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{(x_{i+2} - x_i)}$$

$$P_n = f_0 + s\cdot\Delta f_0 + \frac{s\cdot(s-1)}{2!}\Delta^2 f_0 + \frac{s\cdot(s-1)(s-2)}{3!}\Delta^3 f_0 + \ldots$$
$$+ \frac{s\cdot(s-1)\cdots\cdots(s-n+1)}{n}\Delta n f_0$$

$$s = (x - x_0)/h \;;\; h = \Delta x$$

$$\Delta f_0 = f_1 - f_0$$

$$\Delta^n f_0 = \Delta^{n-1} f_1 - \Delta^{n-1} f_0$$

$$f_n(x)=\sum_{i=0}^{n} L_i(x)\,f(x_i) \qquad L_i(x)=\prod_{j=0,\,j\neq i}^{n}\frac{x-x_j}{x_i-x_j}=\frac{(x-x_0)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$$

$$f_1(x)=L_0(x)\,f(x_0)+L_1(x)\,f(x_1)$$

Forward difference:

$$f'(x)=\frac{f(x+h)-f(x)}{h}+O(h)$$

$$f'(x)=\frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}+O(h^2)$$

$$f''(x)=\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}+O(h)$$

$$f''(x)=\frac{f(x+3h)+4f(x+2h)-5f(x+h)+2f(x)}{h^2}+O(h^2)$$

Backward difference:

$$f'(x)=\frac{f(x)-f(x-h)}{h}+O(h)$$

$$f'(x)=\frac{3f(x)-4f(x-h)+f(x-2h)}{2h}+O(h^2)$$

$$f''(x)=\frac{f(x)-2f(x-h)+f(x-2h)}{h^2}+O(h)$$

$$f''(x)=\frac{2f(x)-5f(x-h)+4f(x-2h)-f(x-3h)}{h^2}+O(h^2)$$

Central difference

$$f'(x)=\frac{f(x+h)-f(x-h)}{2h}+O(h^2)$$

$$f'(x)=\frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}+O(h^4)$$

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}+O(h^2)$$

$$f''(x)=\frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}+O(h^4)$$

$$\int_a^b f(x)dx\cong\frac{h}{2}(f(a)+f(b))-\frac{f''(\eta)}{12}h^3 \qquad \int_a^b f(x)dx\cong(b-a)\frac{f(x_0)+2\sum_{i=1}^{n-1}f(x_i)+f(x_n)}{2n}$$

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}\big(f(x_0)+4f(x_1)+f(x_2)\big)-\frac{f''''(\eta)}{90}h^5=(b-a)\frac{\big(f(x_0)+4f(x_1)+f(x_2)\big)}{6}$$

$$\int_a^b f(x)dx=(b-a)\frac{f(x_0)+4\sum_{i=1,3,5}^{n-1}f(x_i)+2\sum_{j=2,4,6}^{n-2}f(x_j)+f(x_n)}{3n}$$

$$=\frac{h}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{2n-3})+2f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n}))$$

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{8}\big(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\big)-\frac{3f''''(\eta)}{80}h^5=(b-a)\frac{\big(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\big)}{8}$$

$$I_{n+2} \cong I_n(h_2) + \left[\frac{1}{(2)^n - 1}\right]\left[I_n(h_2) - I_n(h_1)\right] \quad \text{or} \quad I_{j,k} \cong \frac{\left[4^{k-1} \cdot I_{j+1,k-1}\right] - I_{j,k-1}}{4^{k-1} - 1}$$

$$\int_a^b f(x)dx \cong \sum_{i=1}^{n} c_i f(t_i) \qquad x = \frac{(b-a)}{2}t + \frac{(a+b)}{2} \quad ; \quad dx = \frac{(b-a)}{2}dt$$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t + (a+b)}{2}\right)\frac{(b-a)}{2}dt = \frac{(b-a)}{2}\int_{-1}^{1} g(t)dt \qquad y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$y_{i+1}^0 = y_i + f\left(x_i, y_i\right)h$$

$$y_{i+1} = y_i + \frac{f\left(x_i, y_i\right) + f\left(x_{i+1}, y_{i+1}^0\right)}{2}h$$

$$y_{i+1/2}^0 = y_i + f\left(x_i, y_i\right)\frac{h}{2}$$

$$y'_{i+1/2} = f\left(x_{i+1/2}, y_{i+1/2}\right)$$

$$y_{i+1} = y_i + f\left(x_{i+1/2}, y_{i+1/2}\right)h$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i) \qquad k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) \qquad k_4 = f\left(x_i + h, y_i + k_3 h\right)$$

$$y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i-k} + O(h^{n+1}) \qquad y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O(h^{n+1})$$

Coefficients for Adams − Bashforth predictors

Order	β_0	β_1	β_2	β_3	β_4	β_5
1	1					
2	3/2	-1/2				
3	23/12	-16/12	5/12			
4	55/24	-59/24	37/24	-9/24		
5	1901/720	-2774/720	2616/720	-1274/720	251/720	
6	4277/720	-7923/720	9982/720	-7298/720	2877/720	-475/720

Coefficients for Adams − Moulton correctors

Order	β_0	β_1	β_2	β_3	β_4	β_5
2	1/2	1/2				
3	5/12	8/12	-1/12			
4	9/24	19/24	-5/24	1/24		
5	251/720	646/720	-264/720	106/720	-19/720	
6	475/1440	1427/1440	-798/1440	482/1440	-173/1440	27/1440

Weighing factors c and function integration points t for Gaussian Quadrature

No. of Points N	Weighing Factors c_i	Function Arguments t_i
1	$c_0 = 2$	$t_0 = 0$
2	$c_0 = 1.0000000$ $c_1 = 1.0000000$	$t_0 = -0.577350269$ $t_1 = 0.577350269$
3	$c_0 = 0.5555556$ $c_1 = 0.8888889$ $c_2 = 0.5555556$	$t_0 = -0.774596669$ $t_1 = 0$ $t_2 = 0.774596669$
4	$c_0 = 0.3478548$ $c_1 = 0.6521452$ $c_2 = 0.6521452$ $c_3 = 0.3478548$	$t_0 = -0.861136312$ $t_1 = -0.339981044$ $t_2 = 0.339981044$ $t_3 = 0.861136312$
5	$c_0 = 0.2369269$ $c_1 = 0.4786287$ $c_2 = 0.5688889$ $c_3 = 0.4786287$ $c_4 = 0.2369269$	$t_0 = -0.906179846$ $t_1 = -0.538469310$ $t_2 = 0$ $t_3 = 0.538469310$ $t_4 = 0.906179846$
6	$c_0 = 0.1713245$ $c_1 = 0.3607616$ $c_2 = 0.4679139$ $c_3 = 0.4679139$ $c_4 = 0.3607616$ $c_5 = 0.1713245$	$t_0 = -0.932469514$ $t_1 = -0.661209386$ $t_2 = -0.238619186$ $t_3 = 0.238619186$ $t_4 = 0.661209386$ $t_5 = 0.932469514$