7 Analytic Trigonometry

7.1 Trigonometric Identities

Let's begin by listing the identities we already know.

Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Even/Odd Identities:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

Cofunction Identities:

$$\begin{array}{ll}
\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta & \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \\
\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta & \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta \\
\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta & \csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta
\end{array}$$

<u>Examples:</u>

1. Simplify $\cos(t) + \tan(t)\sin(t)$

   <u>*solution:*</u>

$$\begin{aligned}
\cos(t) + \tan(t)\sin(t) &= \cos(t) + \frac{\sin(t)}{\cos(t)}\sin(t) \\
&= \cos(t) + \frac{\sin^2(t)}{\cos(t)} \\
&= \frac{\cos^2(t) + \sin^2(t)}{\cos(t)} \\
&= \frac{1}{\cos(t)} \\
&= \sec(t)
\end{aligned}$$

2. Simplify $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$.

solution:

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} &= \frac{\sin\theta(1+\sin\theta) + \cos^2\theta}{\cos\theta(1+\sin\theta)} \\
&= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta(1+\sin\theta)} \\
&= \frac{\sin\theta + 1}{\cos\theta(1+\sin\theta)} \\
&= \frac{1}{\cos\theta} = \sec\theta
\end{aligned}$$

3. Verify the identity

$$\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$$

solution: To verify identities we have to transform one side until it looks like the other - do not move terms from side to side.

$$\begin{aligned}
LHS &= \cos\theta(\sec\theta - \cos\theta) \\
&= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) \\
&= \cos\theta\left(\frac{1-\cos^2\theta}{\cos\theta}\right) \\
&= 1 - \cos^2\theta \\
&= \sin^2\theta = RHS
\end{aligned}$$

4. Verify the identity

$$2\tan\theta\sec\theta = \frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta}$$

solution: We work with the right hand side

$$\begin{aligned}
RHS &= \frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta} \\
&= \frac{1+\sin\theta - (1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} \\
&= \frac{2\sin\theta}{1-\sin^2\theta} \\
&= \frac{2\sin\theta}{\cos^2\theta} \\
&= 2\tan\theta\sec\theta = LHS
\end{aligned}$$

5. Verify the identity

$$\frac{\cos u}{1-\sin u} = \sec u + \tan u$$

solution: This one is a little trickier. We start with the right hand side:

$$\begin{aligned} RHS &= \sec u + \tan u \\ &= \frac{1}{\cos u} + \frac{\sin u}{\cos u} \\ &= \frac{1+\sin u}{\cos u} \end{aligned}$$

Now we appear to be stuck. We want this to look like the LHS which has $1 - \sin u$ in the denominator, so we could try multiplying the RHS by this on the top and bottom to see what happens:

$$\begin{aligned} RHS &= \frac{1+\sin u}{\cos u} \cdot \frac{1-\sin u}{1-\sin u} \\ &= \frac{(1+\sin u)(1-\sin u)}{\cos u(1-\sin u)} \\ &= \frac{1-\sin^2 u}{\cos u(1-\sin u)} \\ &= \frac{\cos^2 u}{\cos u(1-\sin u)} \\ &= \frac{\cos u}{1-\sin u} = LHS \end{aligned}$$

6. Verify the identity

$$\frac{1+\cos\theta}{\cos\theta} = \frac{\tan^2\theta}{\sec\theta - 1}$$

solution: We should start on the right hand side and convert to sines and coses:

$$
\begin{aligned}
RHS &= \frac{\tan^2\theta}{\sec\theta - 1} \\
&= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos\theta} - 1} \\
&= \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1-\cos\theta}{\cos\theta}} \\
&= \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\cos\theta}{1-\cos\theta} \\
&= \frac{(1-\cos^2\theta)\cos\theta}{\cos^2\theta(1-\cos^\theta)} \\
&= \frac{(1-\cos\theta)(1+\cos\theta)}{\cos\theta(1-\cos^\theta)} \\
&= \frac{1+\cos\theta}{\cos\theta} = RHS
\end{aligned}
$$

7.2 Addition and Subtraction Formulas

Formulas for sine:

$$
\begin{aligned}
\sin(s+t) &= \sin s\cos t + \cos s\sin t \\
\sin(s-t) &= \sin s\cos t - \cos s\sin t
\end{aligned}
$$

Formulas for cosine:

$$
\begin{aligned}
\cos(s+t) &= \cos s\cos t - \sin s\sin t \\
\cos(s-t) &= \cos s\cos t + \sin s\sin t
\end{aligned}
$$

Formulas for tangent:

$$
\begin{aligned}
\tan(s+t) &= \frac{\tan s + \tan t}{1 - \tan s\tan t} \\
\tan(s-t) &= \frac{\tan s - \tan t}{1 + \tan s\tan t}
\end{aligned}
$$

Example: Find $\cos(75^\circ)$ and $\cos\left(\frac{\pi}{12}\right)$

solution:

$$
\begin{aligned}
\cos(75^\circ) &= \cos(45^\circ + 30^\circ) \\
&= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ) \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \frac{1}{2}\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

<u>Expressions of the form $A\sin x + B\cos x$</u>

We can use the sum and difference identities to rewrite expressions of the form $A\sin x + B\cos x$ as something simpler. The trick is to simultaneously transform the A into something that looks like $\sin\phi$ and transform the B into something that looks like $\cos\phi$. Then we can use a sum identity. We do this by imagining a point in the xy-plane with coordinates (A, B). If ϕ is the angle between the line connecting (A, B) with the origin and the x-axis, then

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}} \qquad \sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$

Thus we can rewrite, using the sum identity:

$$\begin{aligned}
A\sin x + B\cos x &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin x + \frac{B}{\sqrt{A^2+B^2}}\cos x\right) \\
&= \sqrt{A^2+B^2}(\cos\phi\sin x + \sin\phi\cos x) \\
&= \sqrt{A^2+B^2}\sin(x+\phi)
\end{aligned}$$

<u>Example:</u> Express $3\sin x + 4\cos x$ in the form $k\sin(x+\phi)$

<u>*solution:*</u> Here, $k = \sqrt{A^2+B^2} = \sqrt{3^2+4^2} = 5$ also, $\sin\phi = \frac{4}{5}$, $\cos\phi = \frac{3}{5}$ so $\phi = 0.927$ radians.

Thus,

$$3\sin x + 4\cos x = 5\sin(x+0.927)$$

7.3 Double-Angle, Half-Angle, and Product-sum Formulas

We state the double-angle identites:

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Example: If $\cos x = -\frac{2}{3}$ and x is in quadrant II, find $\cos 2x$ and $\sin 2x$.

solution:

$$\cos 2x = 2\cos^2 x - 1 = 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}$$

$$\sin 2x = 2\sin x \cos x$$

Since x is in quadrant II, we can sub in $\sin x = \sqrt{1 - \cos^2 x}$:

$$\sin 2x = 2\left(-\frac{2}{3}\right)\sqrt{1 - \left(-\frac{2}{3}\right)^2} = -\frac{4}{3}\sqrt{1 - \frac{4}{9}} = -\frac{4}{3}\sqrt{\frac{5}{9}} = -\frac{4\sqrt{5}}{9}$$

Formulas for Lowering Powers

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Half-angle formulas

$$\sin\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\left(\frac{u}{2}\right) = \frac{1 - \cos u}{\sin u}$$

The $\pm$ is chosen in the first two depending on what quadrant $\frac{u}{2}$ is in.

Example: Find the exact value of $\sin 22.5^\circ$.

solution: Since 22.5° is half of 45°, we use the half-angle formula for sin with $u = 45^\circ$.

Since $22.5°$ is in the first quadrant, we choose the + sign.

$$\begin{aligned}
\sin 22.5° &= \sin\left(\frac{45°}{2}\right) \\
&= \sqrt{\frac{1-\cos 45°}{2}} \\
&= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2-\sqrt{2}}{4}}
\end{aligned}$$

<u>Example:</u> Fine $\tan\left(\frac{u}{2}\right)$ if $\sin u = \frac{2}{5}$ and u is in quadrant II.

<u>*solution:*</u> From the formula,

$$\tan\left(\frac{u}{2}\right) = \frac{1-\cos u}{\sin u}$$

We know $\sin u = \frac{2}{5}$, and from the usual pythagorean identity we know

$$\cos u = \pm\sqrt{1-\sin^2 u} = -\sqrt{1-\sin^2 u}$$

We choose the negative because cos is negative in the second quadrant. Hence

$$\begin{aligned}
\tan\left(\frac{u}{2}\right) &= \frac{1-\cos u}{\sin u} \\
&= \frac{1+\sqrt{1-\sin^2 u}}{\sin u} \\
&= \frac{1+\sqrt{1-(\frac{2}{5})^2}}{2/5} \\
&= \frac{1+\sqrt{\frac{21}{25}}}{2/5} \\
&= \frac{5}{2}\left(\frac{5+\sqrt{21}}{5}\right) \\
&= \frac{5+\sqrt{21}}{2}
\end{aligned}$$

Product-to-Sum Formulas

$$\begin{aligned}
\sin u \cos v &= \frac{1}{2}[\sin(u+v) + \sin(u-v)] \\
\cos u \sin v &= \frac{1}{2}[\sin(u+v) - \sin(u-v)] \\
\cos u \cos v &= \frac{1}{2}[\cos(u+v) + \cos(u-v)] \\
\sin u \sin v &= \frac{1}{2}[\sin(u-v) - \sin(u+v)]
\end{aligned}$$

Example: Express $\sin 3x \sin 5x$ as a sum of trig functions.

solution: We use the fourth product-to-sum formula:

$$\sin 3x \sin 5x = \frac{1}{2}[\sin(3x-5x) - \sin(3x+5x)] = \frac{1}{2}(\cos(-2x) - \cos 8x) = \frac{1}{2}(\cos(2x) - \cos 8x)$$

Sum-to-Product Formulas

$$\begin{aligned}
\sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\sin x - \sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\
\cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\
\cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)
\end{aligned}$$

Example: Write $\sin 7x + \sin 3x$ as a product.

solution: We use the first formula:

$$\sin 7x + \sin 3x = 2\sin\left(\frac{7x+3x}{2}\right)\cos\left(\frac{7x-3x}{2}\right) = 2\sin 5x \cos 2x$$

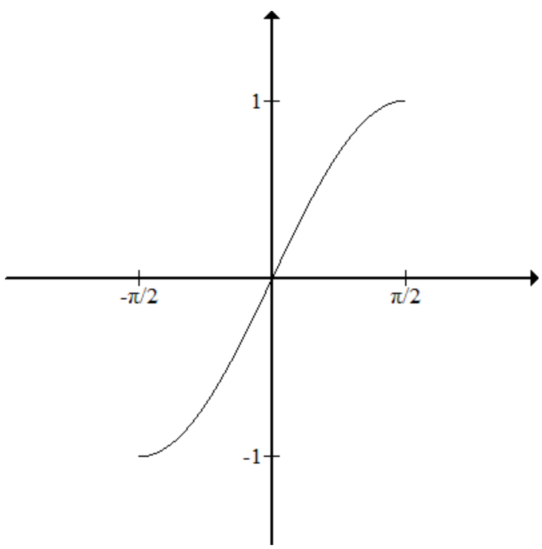
7.4 Inverse Trigonometric Functions

From looking at the graphs of the trig functions, we see that they fail the horizontal line test spectacularly. However, if you restrict their domain, you can find an inverse for these functions on this domain only.

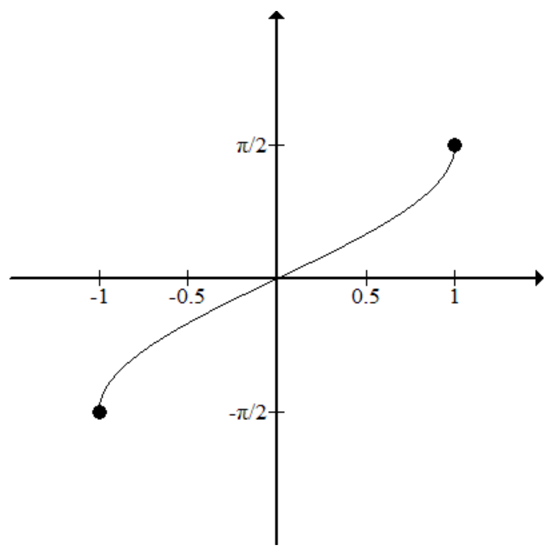
Inverse Sine

The function sin is one-to-one when restricted to $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Thus there exists a function $\sin^{-1}$ or arcsin

$$\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$\sin x$



$\arcsin x (= \sin^{-1} x)$

The arcsin function satisfies

$$\arcsin(x) = y \Leftrightarrow \sin(y) = x$$
$$\arcsin(\sin x) = x \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\arcsin x) = x \text{ if } -1 \leq x \leq 1$$

Examples: Find (a)$\sin^{-1}(\frac{1}{2})$, (b)$\arcsin(-\frac{1}{2})$, (c)$\sin^{-1}(\frac{3}{2})$ and (d)$\cos(\sin^{-1}(\frac{3}{5}))$.

solution:

(a) We know that $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

(b) Likewise, $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$, so $\arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$.

(c) We know $\sin x$ is never $\frac{3}{2}$ (it is never greater than 1), so $\sin^{-1}(\frac{3}{2})$ is undefined.

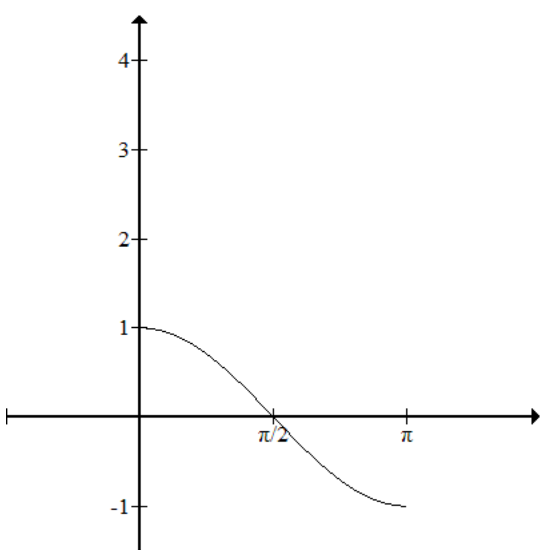
(d) For $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $\cos(u) = \sqrt{1 - \sin^2 u}$, so

$$\begin{aligned}\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right) &= \sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{3}{5}\right)\right)} \\ &= \sqrt{1 - \left(\frac{3}{5}\right)^2} \\ &= \sqrt{1 - \frac{9}{25}} \\ &= \sqrt{\frac{16}{25}} \\ &= \frac{4}{5}\end{aligned}$$

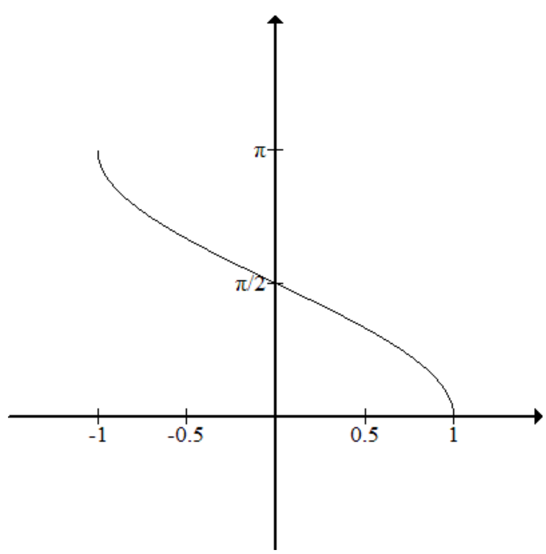
<u>Inverse Cosine</u>

The function cos is one-to-one when restricted to $\cos : [0, \pi] \to [-1, 1]$. Thus there exists a function $\cos^{-1}$ or arccos

$$\cos^{-1} : [-1, 1] \to [0, \pi]$$



$\cos x$



$\arccos x (= \cos^{-1} x)$

The arccos function satisfies

$$\arccos(x) = y \Leftrightarrow \cos(y) = x$$

$$\arccos(\cos x) = x \text{ if } 0 \leq x \leq \pi$$

$$\cos(\arccos x) = x \text{ if } -1 \leq x \leq 1$$

Examples: Find (a) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\cos^{-1}(0)$.

solution:

(a) $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, thus $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

(b) $\cos(\frac{\pi}{2}) = 0$, thus $\cos^{-1}(0) = \frac{\pi}{2}$

Examples:

1. Show that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

   solution: $\sin u = \sqrt{1-\cos^2(u)}$ for $u \in [0, \pi]$. Thus

   $$\sin(\cos^{-1}(x)) = \sqrt{1-\cos^2(\cos^{-1}(x))} = \sqrt{1-x^2}$$

2. Show that $\tan(\cos^{-1}(x)) = \frac{\sqrt{1-x^2}}{x}$.

   solution: $\tan(u) = \frac{\sin u}{\cos u}$, so

   $$\tan(\cos^{-1}(x)) = \frac{\sin(\cos^{-1}(x))}{\cos(\cos^{-1}(x))} = \frac{\sqrt{1-x^2}}{x}$$

3. Show that $\sin(2\cos^{-1}(x)) = 2x\sqrt{1-x^2}$

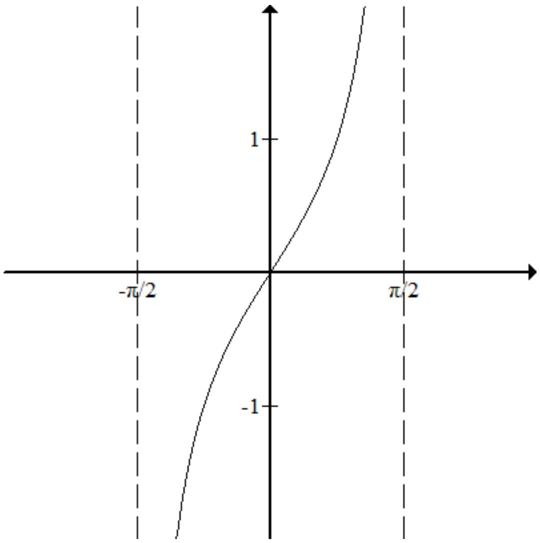
   solution: $\sin 2u = 2\sin u \cos u$, so

   $$\sin(2\cos^{-1}(x) = 2\sin(\cos^{-1}(x))\cos(\cos^{-1}(x)) = 2\sqrt{1-x^2}x = 2x\sqrt{1-x^2}$$

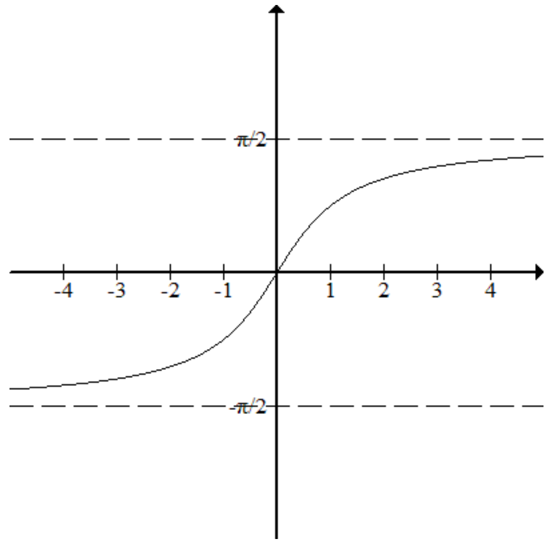
Inverse Tangent

The function tan is one-to-one when restricted to $\tan : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$. Thus there exists a function $\tan^{-1}$ or arctan

$$\tan^{-1} : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$\tan x$



$\arctan x (= \tan^{-1} x)$

The arctan function satisfies

$$\arctan(x) = y \Leftrightarrow \tan(y) = x$$
$$\arctan(\tan x) = x \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\tan(\arctan x) = x \text{ if } x \in \mathbb{R}$$

Examples: Find (a) $\tan^{-1}(1)$, (b) $\tan^{-1}\sqrt{3}$ and (c)$\arctan(-20)$

solution:

(a) $\tan(\frac{\pi}{4}) = 1$, thus $\tan^{-1}(1) = \frac{\pi}{4}$.

(b) $\tan(\frac{\pi}{3}) = \sqrt{3}$, thus $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

(c) Using a calculator, we find $\arctan(-20) \approx 1.52084$

Other Inverse Trig Functions

The other trig functions $\cot, \csc$ and $\sec$ also have inverses when restricted to suitable domains, namely $\cot^{-1}, \csc^{-1}$ and $\sec^{-1}$. You don't need to worry about graphing these. Just keep in mind:

$$\cot^{-1} \neq \frac{1}{\tan^{-1}}$$
$$\csc^{-1} \neq \frac{1}{\sin^{-1}}$$
$$\sec^{-1} \neq \frac{1}{\cos^{-1}}$$

7.5 Trigonometric Equations

One frequently has to solve equations involving trig functions. Sometimes the values of x you look at are restricted, while others you are asked to find all the values of x that make a given equation true. In the latter case, there are usually an infinite number of solutions whose form depends on the period of the trig functions involved.

Examples:

1. Solve

   (a) $\tan^2(x) - 3 = 0$

   (b) $\sin(x) = \cos(x)$

   (c) $1 + \sin x = 2\cos^2(x)$

   (d) $\sin 2x - \cos x = 0$

   (e) $\cos(x) + 1 = \sin x$ with $t \in [0, 2\pi]$

   solution:

   (a) $\tan^2(x) - 3 = 0 \Rightarrow \tan(x) = \pm\sqrt{3}$.
   $\tan(x) = \sqrt{3}$ when $x = \frac{\pi}{3}$. But tan has period π, so

   $$\tan(x) = \sqrt{3} \Rightarrow x = \frac{\pi}{3} + k\pi \quad k \in \mathbb{Z}$$

   Likewise,

   $$\tan(x) = -\sqrt{3} \Rightarrow x = -\frac{\pi}{3} + k\pi \quad k \in \mathbb{Z}$$

   Thus the full set of solutions is

   $$S = \left\{-\frac{\pi}{3} + k\pi, x = \frac{\pi}{3} + k\pi \mid k \in \mathbb{Z}\right\}$$

   (b) If $\sin(x) = \cos(x)$, $\cos(x) \neq 0$ because sin and cos are not 0 in the same places. Thus we can divide both sides by cos and get

   $$\frac{\sin(x)}{\cos(x)} = 1 \Rightarrow \tan(x) = 1$$

   This happens when $x = \frac{\pi}{4}$. Once again, tan has a period of π so

   $$x = \frac{\pi}{4} + k\pi \quad k \in \mathbb{Z}$$

   (c) We can get the equation $1 + \sin x = 2\cos^2(x)$ into an easier form to deal with by subbing in $1 - \sin^2 x$ for $\cos^2 x$:

   $$\begin{aligned} 1 + \sin x &= 2\cos^2(x) \\ 1 + \sin x &= 2 - 2\sin^2 x \\ 2\sin^2 x + \sin x - 1 &= 0 \end{aligned}$$

This is a quadratic in $\sin x$ that factors as

$$(2\sin x - 1)(\sin x + 1) = 0$$

Which means either $2\sin x - 1 = 0$ or $\sin x + 1 = 0$. The first gives us that $\sin x = \frac{1}{2}$, and the second gives $\sin x = -1$.

- $\sin x = -1 \Rightarrow x = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}$
- $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$ or $x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$

(d) We use a double angle identity on $\sin 2x - \cos x = 0$ to get

$$2\sin x \cos x - \cos x = 0$$

$$\cos x(2\sin x - 1) = 0$$

Hence either

- $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$ or $x = \frac{3\pi}{2} + 2k\pi, k \in \mathbb{Z}$.
- $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z}$ or $x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}$.

(e) Here we have to be a little trickier and square both sides

$$\begin{aligned}
\cos x + 1 &= \sin x \\
(\cos x + 1)^2 &= 1 - \cos^2 x \\
\cos^2 x + 2\cos x + 1 &= 1 - \cos^2 x \\
2\cos^2 x + 2\cos x &= 0 \\
2\cos x(\cos x - 1) &= 0
\end{aligned}$$

Hence either $\cos x = 0$ or $\cos x = -1$. Between 0 and 2π, the first only happens at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and the second happens at $-\pi$. Hence the three solutions are

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi$$

2. Consider the equation $2\sin(3x) - 1 = 0$.

   (a) Find all the solutions to the equation.

   (b) Find all the solutions to the equation in the interval $[0, 2\pi)$.

   solution:

   (a) The equation rearranges to $\sin(3x) = \frac{1}{2}$. As we've seen before, this means that

$$3x = \frac{\pi}{6} + 2k\pi, k \in \mathbb{Z} \text{ or } 3x = \frac{5\pi}{6} + 2k\pi, k \in \mathbb{Z}.$$

$$x = \frac{\pi}{18} + \frac{2k\pi}{3}, k \in \mathbb{Z} \text{ or } x = \frac{5\pi}{18} + \frac{2k\pi}{3}, k \in \mathbb{Z}.$$

(b) We solve the inequalities

$$0 \le \frac{\pi}{18} + \frac{2k\pi}{3} < 2\pi$$
$$0 \le \frac{1}{18} + \frac{2k}{3} < 2$$
$$-\frac{1}{18} \le \frac{2k}{3} < 2 - \frac{1}{18} = \frac{35}{18}$$
$$-\frac{1}{6} \le 2k < \frac{35}{6}$$
$$-\frac{1}{12} \le k < \frac{35}{12} \approx 2.91$$

The ks in this range are $k = 0$, $k = 1$ and $k = 2$. Hence

$$x = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}$$

For the other solutions we have

$$0 \le \frac{5\pi}{18} + \frac{2k\pi}{3} < 2\pi$$
$$0 \le \frac{5}{18} + \frac{2k}{3} < 2$$
$$-\frac{5}{18} \le \frac{2k}{3} < 2 - \frac{5}{18} = \frac{31}{18}$$
$$-\frac{5}{6} \le 2k < \frac{31}{6}$$
$$-\frac{5}{12} \le k < \frac{31}{12} \approx 2.58$$

The ks in this range are $k = 0$, $k = 1$ and $k = 2$. Hence

$$x = \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}$$

3. Consider the equation $\sqrt{3}\tan(\frac{x}{2}) - 1 = 0$.

(a) Find all the solutions of the equation.

(b) Find all the solutions in the interval $[0, 4\pi)$.

solution:

(a) This rearranges to $\tan(\frac{x}{2}) = \frac{1}{\sqrt{3}}$ This gives us

$$\frac{x}{2} = \frac{\pi}{6} + k\pi, k \in \mathbb{Z}$$
$$x = \frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}$$

(b) As before, we find the ks we need:

$$0 \leq \frac{\pi}{3} + 2k\pi < 4\pi$$

$$0 \leq \frac{1}{3} + 2k < 4$$

$$-\frac{1}{3} \leq 2k < 4 - \frac{1}{3} = \frac{11}{3}$$

$$-\frac{1}{6} \leq k < \frac{11}{6} \approx 1.83$$

The ks in this range are $k = 0$ and $k = 1$. Hence $x = \frac{\pi}{3}$ or $x = \frac{7\pi}{3}$.

4. Solve the equation $\tan^2 x - \tan x - 2 = 0$.

<u>*solution:*</u> This is a quadratic in $\tan x$, so

$$\tan^2 x - \tan x - 2 = 0$$

$$(\tan x - 2)(\tan x + 1) = 0$$

So $\tan x = 2$ or $\tan x = -1$. There isn't a convenient angle with $\tan x = 2$, but we can use $\tan^{-1}$ to write

$$x = \tan^{-1}(2) + k\pi, k \in \mathbb{Z}$$

The second one of course gives us

$$x = -\frac{\pi}{4} + k\pi, k \in \mathbb{Z}$$

5. Solve the equation $3 \sin\theta - 2 = 0$.

<u>*solution:*</u> This rearranges to $\sin\theta = \frac{2}{3}$. Once again there is no easy angle that gives us $\sin\theta = \frac{2}{3}$. We know that sin is positive in the first and second quadrants, and thus the two solutions in $[0, 2\pi)$ are

$$\sin^{-1}\left(\frac{2}{3}\right) \text{ and } \pi - \sin^{-1}\left(\frac{2}{3}\right)$$

Hence the full solution is

$$\theta = \sin^{-1}\left(\frac{2}{3}\right) + 2k\pi \text{ or } \theta = \pi - \sin^{-1}\left(\frac{2}{3}\right) + 2k\pi$$