6 Trigonometric Functions of Angles

6.1 Angle Measures

When we think of measuring angles, the unit we usually think of is **degrees**. A right angle is $90°$, a straight angle is $180°$and so on. When talking about angles in the plane, we measure them counterclockwise from the positive x-axis.

There is another sometimes more convenient unit to measure angles, that of the **radian**.

Definition 6.1 *If a circle of radius 1 is drawn with the vertex on an angle at its center, then the measure of this angle in* **radians** *is the length of the arc that subtends this angle.*

If you draw a right angle with its vertex at the center of the unit circle, the length of the arc it subtends is $\frac{\pi}{2}$ as seen last chapter. Thus a right angle is $\frac{\pi}{2}$ radians.

<u>Relationship between Degrees and Radians</u>

$180° = \pi$ rad

$1 \text{ rad} = \left(\dfrac{180}{\pi}\right)^{\circ}$

$1° = \dfrac{\pi}{180}$ rad.

1. To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

<u>Examples:</u>

1. Express $60°$in radians

   <u>*solution:*</u> $60° = 60\left(\frac{\pi}{180}\right)$ rad $= \frac{\pi}{3}$

2. Express $\frac{\pi}{6}$rad in degrees.

   <u>*solution:*</u> $\frac{\pi}{6}$rad $= \frac{\pi}{6}\left(\frac{180}{\pi}\right) = 30°$

<u>Coterminal Angles</u>

Definition 6.2 *Two angles are* **coterminal** *if they differ by a multiple of 2π radians (or 360°).*

Thus when measured counterclockwise from the x-axis, coterminal angles "end up in the same place".

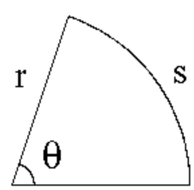
Example: $30°$and $390°$are coterminal since they differ by $360°$.

$\frac{\pi}{2}$ and $\frac{5\pi}{2}$ are coterminal since they differ by 2π.

Arc Length

Radian measure for angle is based on the length of arc subtended on a circle of radius 1. If a circle instead has radius r, then if an angle θ at the origin is measured in radians the length s of the arc spanned by the angle is

$$s = r\theta$$



Examples:

1. Find the length of the arc on a circle radius 10m spanned by an angle of $30°$.

   solution: Note that for our formula to work we need the angle to be in radians.

   $$30° = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$$

   Thus

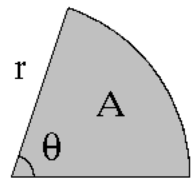
   $$s = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3} \approx 5.23\text{m}$$

2. Find the angle θ if the arc length is 6m and the radius is 4m.

   solution: We use the formula

   $$s = r\theta \Rightarrow \theta = \frac{s}{r} = \frac{6}{4} = \frac{3}{2} \text{ rad}$$

Area of a Circular Sector



The area of the above section is given by

$$A = \frac{1}{2}r^2\theta$$

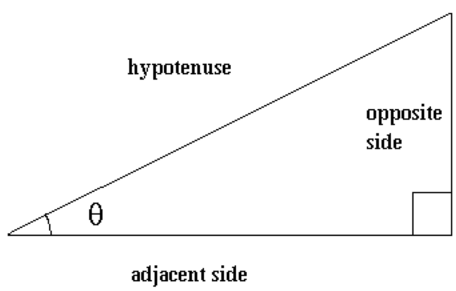
Example: Find the area of the sector of radius 3m and angle $\theta = 60°$

solution: $60° = \frac{\pi}{3}$ radians, so

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2\frac{\pi}{3} = \frac{3\pi}{2}\text{m}^2$$

6.2 Trigonometry of Right Triangles

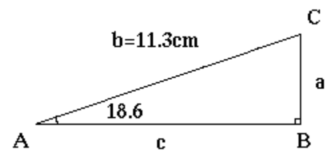
The primary trigonometric relations for a right triangle are



$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Note that if θ is measured in radians, these definitions line up with that of last section.

Example: In the triangle ABC, the angle at B is $90°$, the angle at A is $18.6°$and the side b has length 11.3 cm. Find the measure of the angle at C and find the lengths a and b.



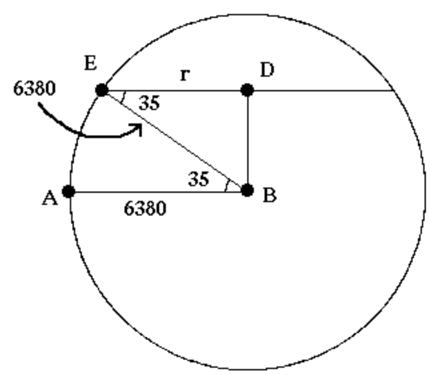
solution: The sum of all angles around a triangle is always $180°$, so the angle at C is $180 - 90 - 18.6 = 71.4°$.

For the side lengths, we notice that

$$\sin(18.6°) = \frac{a}{11.3} \Rightarrow a = 11.3 \cdot \sin(18.6°) = 3.6\text{cm}$$
$$\cos(18.6°) = \frac{c}{11.3} \Rightarrow c = 11.3 \cdot \cos(18.6°) = 10.7\text{cm}$$

Example: Determine the length of the 35th parallel, assuming the earth has radius 6380km.

solution:



From the diagram, we see $AB = BE = 6380$. Also, $\angle DEB = 35°$ by the alternate internal angle rule. Thus,

$$\cos(35°) = \frac{r}{6380} \Rightarrow r = 6380\cos(35°) = 5226$$

The length of the 35th parallel is the circumfrence of the circle with radius r:

$$L = 2\pi r = 2\pi(5226) = 32840\text{km}$$

6.3 Trigonometric Functions of an Angular Variable

Suppose the line between the point (x, y) is at an angle θ measured counterclockwise from the x-axis.

Then by the Pythagorean theorem $r = \sqrt{x^2 + y^2}$. We also have

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

This allows us to take the trig functions of angles greater than 90°.

Example: The line between the origin and the point $(-4, 3)$ makes an angle of θ couter-clockwise from the x-axis. Find $\sin\theta$ and $\cos\theta$.

solution:

$$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

Thus

$$\sin\theta = \frac{y}{r} = \frac{3}{5}$$
$$\cos\theta = \frac{x}{r} = -\frac{4}{5}$$

Example: Find cot(495°)

solution: 495°- 360°= 135°, so 495°is coterminal with 135°. So this angle is in the second quadrant making an angle of 180°-135°= 45°angle with the x-axis. $\cot(45°) = 1$, but cot is negative in quadrant II, so

$$\cot(495°) = -1$$

Example: If θ is in the second quadrant, write $\sin\theta$ and $\tan\theta$ as functions of $\cos\theta$.

solution: We know that $\sin^2\theta + \cos^2\theta = 1$, so

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

Since we are in the second quadrant, sin must be positive, thus we choose the positive one:

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

Now, we know

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and we just wrote sin as a function of cos, so we can sub it in:

$$\tan\theta = \frac{\sqrt{1 - \cos^2\theta}}{\cos\theta}$$