5 Trigonometric Functions

5.1 The Unit Circle

Definition 5.1 *The* **unit circle** *is the circle of radius 1 centered at the origin in the xy-plane:*

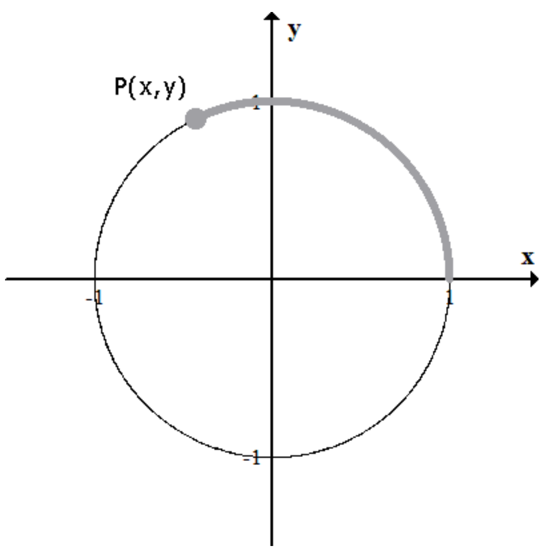
$$x^2 + y^2 = 1$$

<u>Example:</u> The point $P\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle because

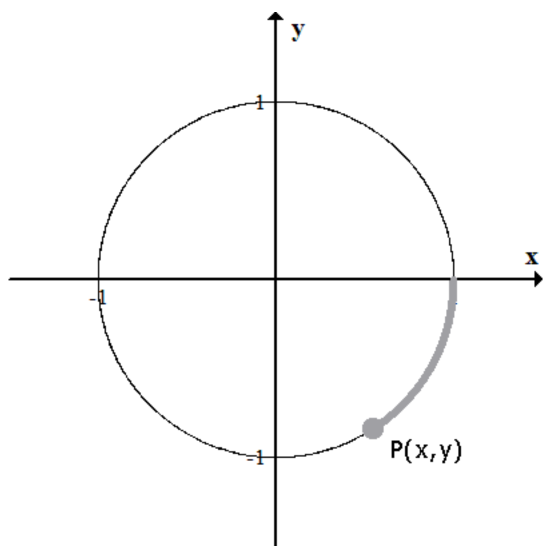
$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = \frac{9}{9} = 1$$

<u>Terminal Points</u>

For any positive real number t, the **terminal point** of t is the point on the unit circle you arrive at if you travel along the unit circle t units from the point $(0, 1)$ in a counterclockwise direction. For negative real numbers, you travel clockwise.



Starting at $(0, 1)$, if $t > 0$ we move in the counterclockwise directon to obtain a terminal point $P(x, y)$. t is the length of the arc.



Starting at $(0, 1)$, if $t < 0$ we move in the clockwise directon.

Recall that the circumfrence of the unit circle is 2π. Thus, we already know some terminal points:

$$\begin{aligned}
t = 2\pi &\quad\longrightarrow\quad P(1,0) \text{ is the terminal point.}\\
t = \pi &\quad\longrightarrow\quad P(-1,0) \text{ is the terminal point.}\\
t = \frac{\pi}{2} &\quad\longrightarrow\quad P(0,1) \text{ is the terminal point.}\\
t = -\pi &\quad\longrightarrow\quad P(-1,0) \text{ is the terminal point.}\\
t = -\frac{\pi}{2} &\quad\longrightarrow\quad P(0,-1) \text{ is the terminal point.}\\
t = -\frac{3\pi}{2} &\quad\longrightarrow\quad P(0,-1) \text{ is the terminal point.}
\end{aligned}$$

Example: Find the terminal point for $t = \frac{\pi}{4}$.

solution: When $t = \frac{\pi}{4}$, $y = x$ so we can use the formula for the circle to find x and y:

$$\begin{aligned}
x^2 + y^2 &= 1\\
x^2 + x^2 &= 1\\
2x^2 &= 1\\
x^2 &= \frac{1}{2}\\
x &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Here we can see that our point is in the first quadrant, so we take $x = y = \frac{1}{\sqrt{2}}$. Thus the terminal point is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Using similar methods, we can prove the following table:

t	Terminal point of t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$

Note: The terminal point for some $t = \alpha$ is the same as $t = \alpha + 2\pi$ for any $\alpha \in \mathbb{R}$. Even further, it will be the same as that of $t = \alpha + 2n\pi$ for any integer n.

Examples: Find the terminal points of the following numbers:

1. $t = -\frac{\pi}{4}$

2. $t = \frac{3\pi}{4}$

3. $t = -\frac{5\pi}{6}$

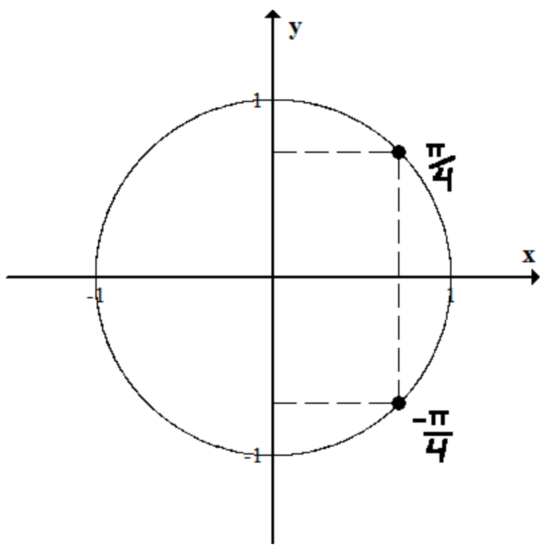
4. $t = \frac{5\pi}{6}$

5. $t = \frac{7\pi}{4}$

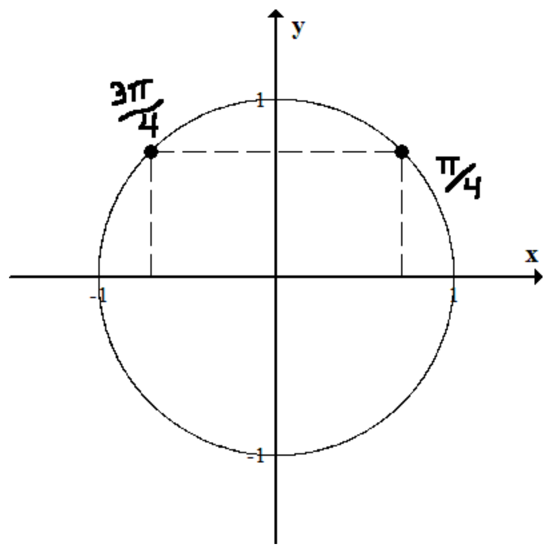
6. $t = \frac{2\pi}{3}$

<u>*solution:*</u>

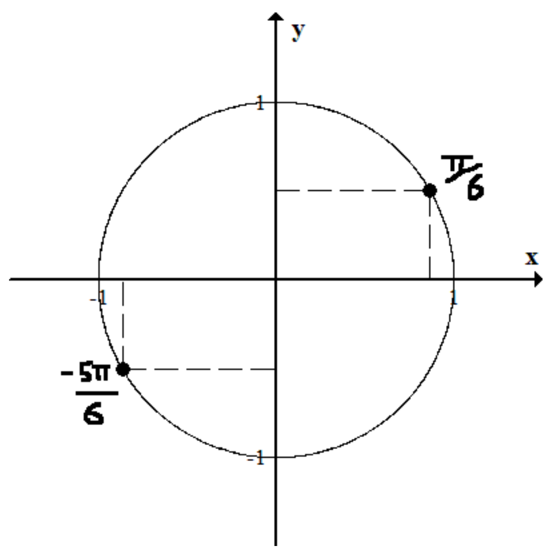
1. If we go clockwise $\frac{\pi}{4}$ we see that the x-coordinate will be the same as if we went counterclockwise $\frac{\pi}{4}$ but the y-coordinate will be the negative of what we would get if we went counterclockwise $\frac{\pi}{4}$. Thus the terminal point is $P\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.



2. This terminal point is the same distance from the x-axis as the terminal point for $\frac{\pi}{4}$ but has negative x-coordinate. Thus the terminal point is $P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



3. Here both the x and y coordinates will be the negatives of that of $\frac{\pi}{6}$ because they are the same distance from the axis along the circle but in the 4th quadrant. Thus the terminal point is $P\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.



4. Here we end up in the second quadrant with the negative of the x-coordinate of $t = \frac{\pi}{6}$ and the same y-coordinate as $t = \frac{\pi}{6}$. Thus the terminal point is $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

5. Here we end up in the fourth quadrant with the same x-coordinate as $t = \frac{\pi}{4}$ and the negative of the y-coordinate of $t = \frac{\pi}{4}$. Thus the terminal point is $P\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

6. Here we end up in the second quadrant with the negative of the x-coordinate as $t = \frac{\pi}{3}$ and the same y-coordinate as $t = \frac{\pi}{3}$. Thus the terminal point is $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

5.2 Trigonometric Functions

Definition 5.2 *Let $t \in \mathbb{R}$ and $P(x, y)$ be the terminal point on the unit circle determined by t. We define*

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{x}{y}$$

$$\csc t = \frac{1}{y} \qquad \sec t = \frac{1}{x} \qquad \cot t = \frac{y}{x}$$

Example: Find the six trig functions of $t = \frac{\pi}{3}$ and $t = \frac{\pi}{2}$.

solution: The critical point of $t = \frac{\pi}{3}$ is $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Thus

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\csc\frac{\pi}{3} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \qquad \sec\frac{\pi}{3} = \frac{1}{\frac{1}{2}} = 2 \qquad \cot\frac{\pi}{3} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

The terminal point of $t = \frac{\pi}{2}$ is $(0, 1)$. Thus,

$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0 \qquad \tan\frac{\pi}{2} = \text{DNE}$$

$$\csc\frac{\pi}{2} = 1 \qquad \sec\frac{\pi}{2} = \text{DNE} \qquad \cot\frac{\pi}{2} = 0$$

Domains of the Trig Functions

Function	Domain
$\sin, \cos$	$\mathbb{R}$
$\tan, \sec$	$\{x \mid x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$
$\cot, \csc$	$\{x \mid x \neq n\pi, n \in \mathbb{Z}\}$

Signs of the Trig Functions

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

Examples:

1. $\cos\frac{\pi}{3} > 0$ because $\frac{\pi}{3}$ is in quadrant I.
2. If $\cos t < 0$ and $\sin t > 0$, then t must be in quadrant II
3. Find $\cos\frac{2\pi}{3}$

   solution: $\frac{2\pi}{3}$ is the same distance along the circle from the x-axis as $\frac{\pi}{3}$, but is in the second quadrant. In the second quadrant cos is negative, so $\cos\frac{2\pi}{3} = -\frac{1}{2}$.
4. Find $\sin\frac{19\pi}{3}$.

   solution: Note that $\frac{19\pi}{3} = \frac{18\pi}{3} + \frac{\pi}{3} = 6\pi + \frac{\pi}{3}$. Since every time you add 2π to t you go around the circle once, $\frac{19\pi}{3}$ will have the same terminal point as $\frac{\pi}{3}$. Thus

$$\sin\frac{19\pi}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

In General:

$$\sin(-t) = -\sin(t) \quad \cos(-t) = \cos(t) \quad \tan(-t) = -\tan(t)$$
$$\csc(-t) = -\csc(t) \quad \sec(-t) = \sec(t) \quad \cot(-t) = -\cot(t)$$

Example: $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = \frac{1}{2}$ while $\cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Fundamental Identities

$$\csc t = \frac{1}{\sin t} \quad \sec t = \frac{1}{\cos t} \quad \cot t = \frac{1}{\tan t}$$
$$\tan t = \frac{\sin t}{\cos t} \quad \cot t = \frac{\cos t}{\sin t}$$
$$\sin^2 t + \cos^2 t = 1$$
$$\tan^2 t + 1 = \sec^2 t$$
$$\cot^2 t + 1 = \csc^2 t$$

Example: If $\cos t = \frac{3}{5}$ and t is in quadrant IV, find the values of all the trig functions at t.

solution: Because of the above identities, we only need to find $\sin t$ and then will be able to use the formulas to get the rest. We use the common identity

$$\sin^2 t + \cos^2 t = 1$$

$$\sin^2 t + \left(\frac{3}{5}\right)^2 = 1$$
$$\sin^2 t + \frac{9}{25} = 1$$
$$\sin^2 t = \frac{16}{25}$$
$$\sin t = \pm\frac{4}{5}$$

Since we are in quadrant IV, sin is negative, so $\sin t = -\frac{4}{5}$.

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{4}{5}} = -\frac{5}{4}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$
$$\cot t = \frac{\cos t}{\sin t} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

5.3 Trigonometric Graphs

Since we have these 6 new functions, we would like to graph them. We first notice a few things:

- The sin function is odd (since $\sin(-t) = -\sin(t)$) so the graph is symmetric about the origin. The cos function is even (since $\cos(-t) = \cos(t)$) so the graph is symmetric about the y-axis.
- The maximum and minimum values of the sine and cosine functions are 1 and -1. This is because those are the max and min values of x and y on the unit circle.
- The sine and cosine functions repeat upon adding 2π to t. That is to say that sin and cos are **periodic with period** 2π:

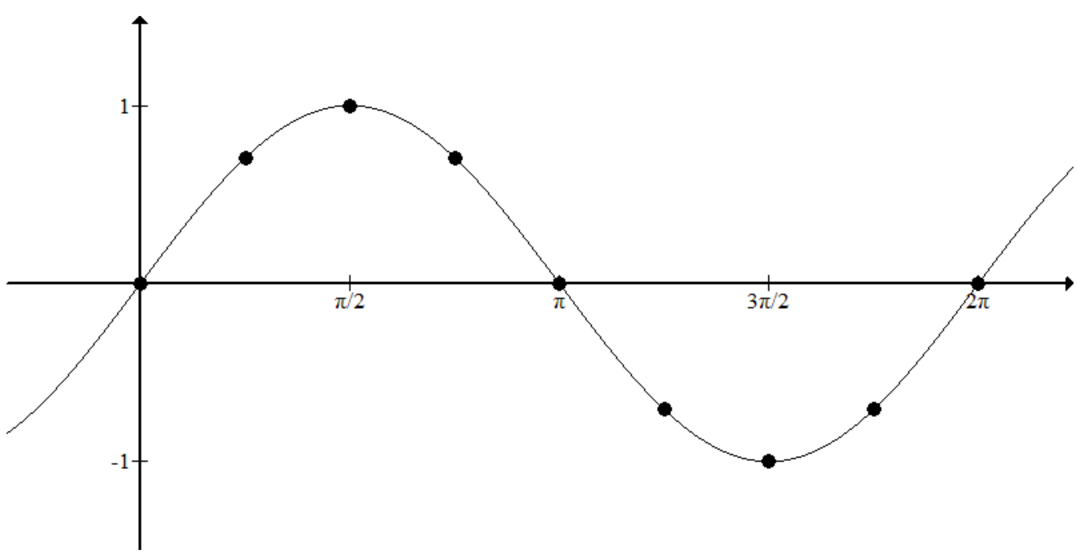
  $$\sin(t + 2\pi) = \sin t$$

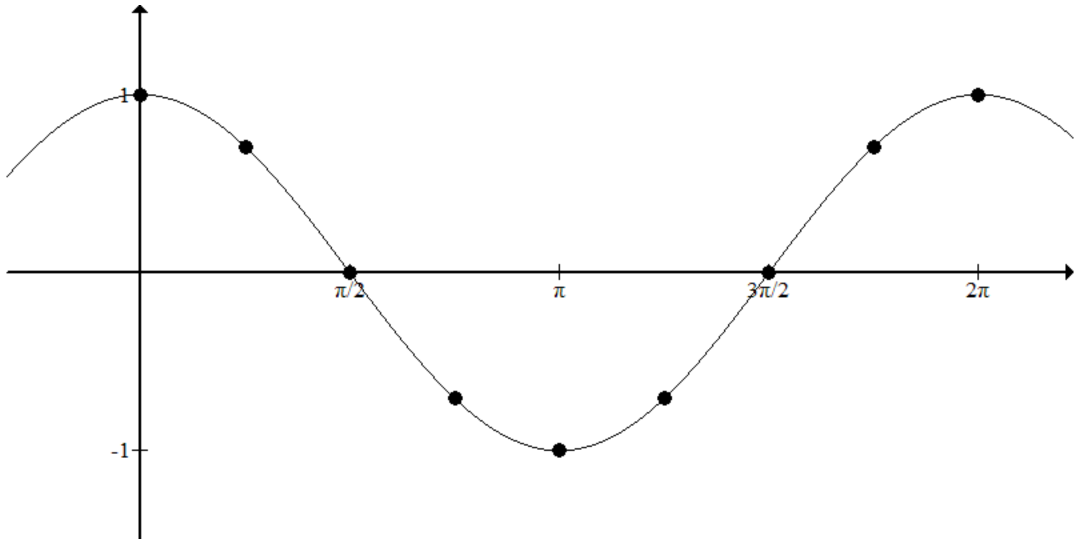
  $$\cos(t + 2\pi) = \cos t$$

  This means that if we can draw the graph of sin or cos from 0 to 2π, we can fill in the rest by repeating the pattern from 0 to 2π.

Example: Let's plot some points for sin and cos.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin t$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0
$\cos t$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1





For all periodic functions and graphs, we define the **period** to be the amount of time it takes the function to repeat one cycle. We also define the **amplitude** to be half the distance between the max and min of the function.

Transformations of sine and cosine functions

Notice that the graphs for sine and cosine are exactly the same except the cos function is shifted to the left by $\frac{\pi}{2}$ from the sin function. Thus

$$\cos(t) = \sin\left(t + \frac{\pi}{2}\right).$$

Conversely, the graph of sin is the same as that of cos shifted to the right by $\frac{\pi}{2}$. Thus

$$\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$$

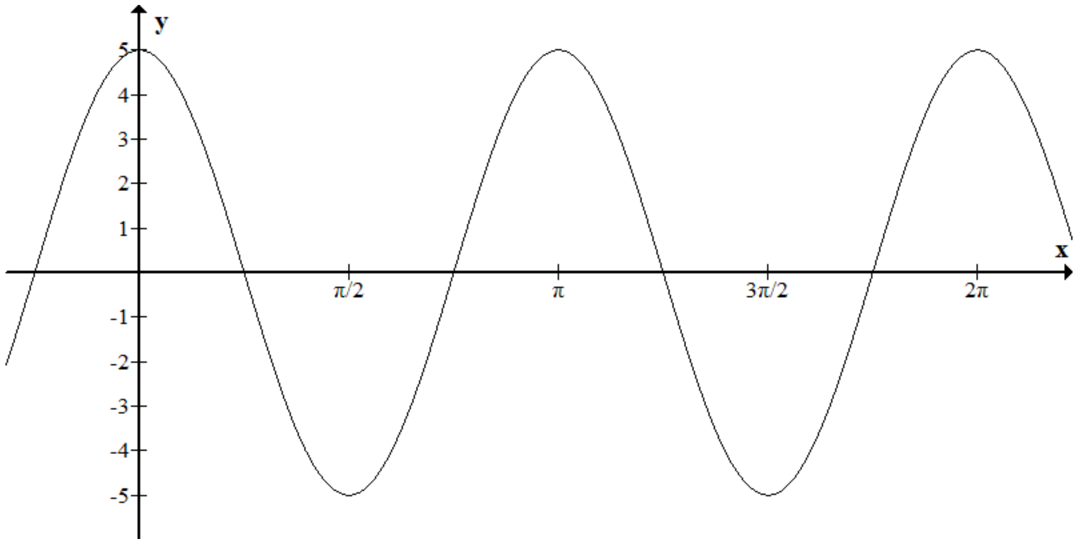
Usually, we will be looking at graphs of the form $y = A\sin Bx$ or $y = A\cos Bx$. In these cases, we have

$$\text{Amplitude } = |A| \text{ and Period } = \frac{2\pi}{B}$$

Examples: Find the amplitude and period of each function, and sketch its graph.

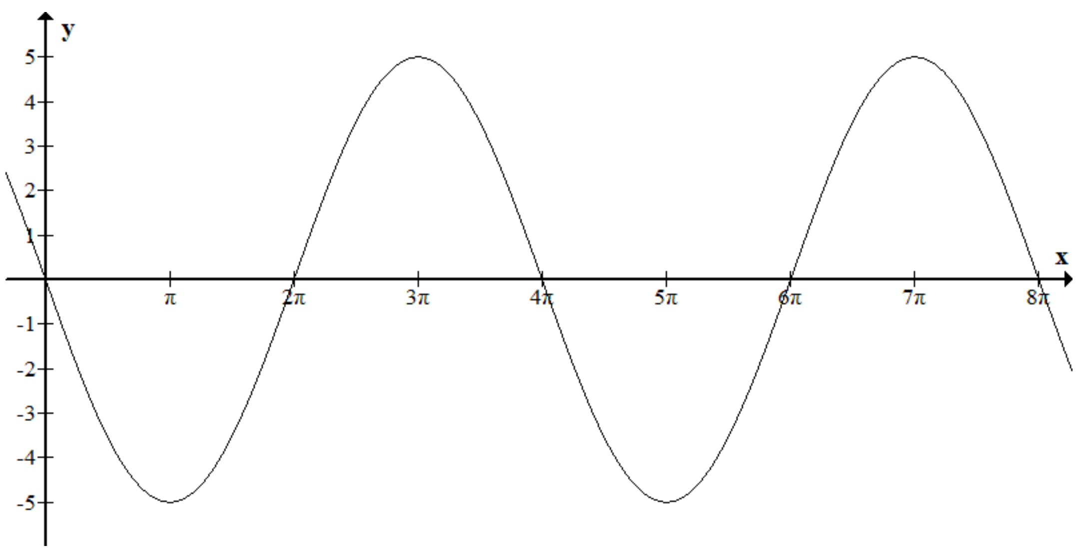
1. $y = 5\cos 2x$.

   solution: The amplitude is $|A| = |5| = 5$ and the period is $\frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$. Thus the graph looks like a cosine graph but with amplitude 5 and period π:



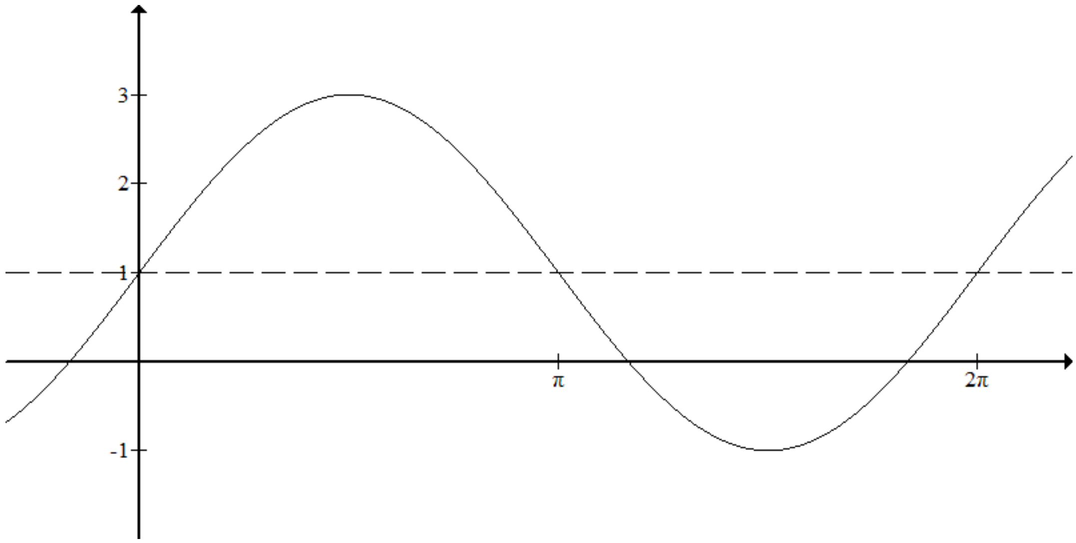
2. $y = -5\sin\left(\frac{t}{2}\right)$

The amplitude is $|A| = |-5| = 5$ and the period is $\frac{2\pi}{|B|} = \frac{2\pi}{1/2} = 4\pi$. Thus the graph looks like a sine graph but with amplitude 5 and period 4π. The negative out front means we reflect the graph about the y-axis:



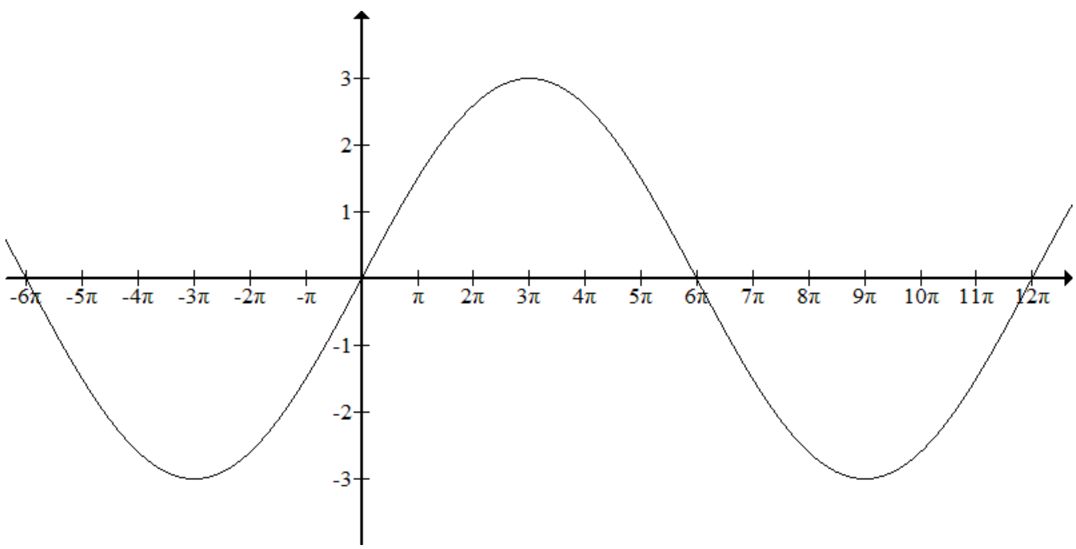
3. $y = 2\sin t + 1$

solution: Here the amplitude is 2 and the period is 2π. The $+1$ gives us a vertical shift of 1:



Examples: Given the following graphs, find the formula of the sine or cosine function which gives them.

1. Consider



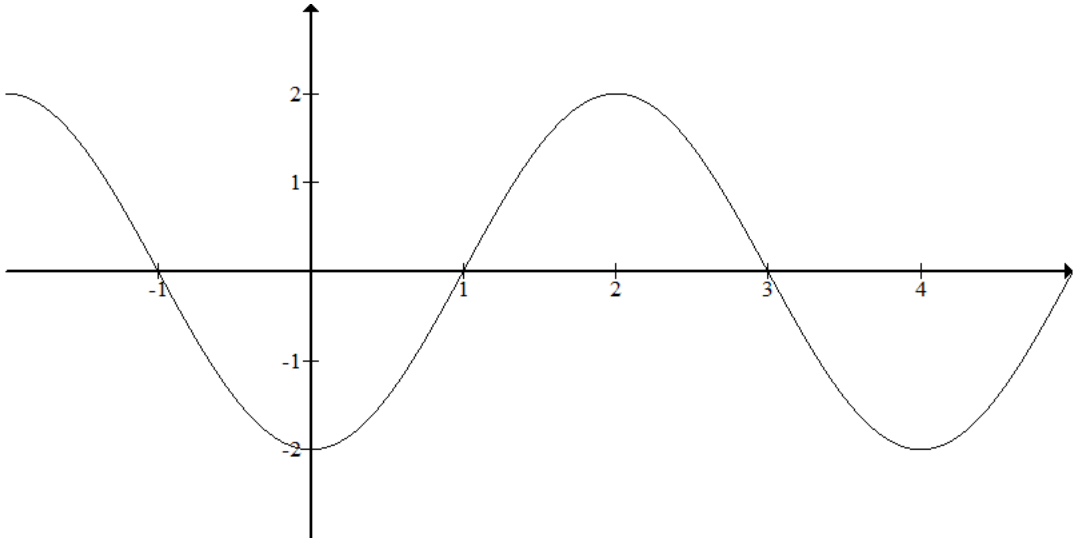
solution: From the graph it looks like the amplitude is 3 because the graph varies between 3 and -3. It also looks like the graph repeats every 12π units. Thus

$$12\pi = \frac{2\pi}{B} \quad \Rightarrow \quad B = \frac{1}{6}$$

Finally, this is a sine graph since it goes through the origin. Hence our equation is

$$y = 3\sin\left(\frac{t}{6}\right)$$

2. Consider



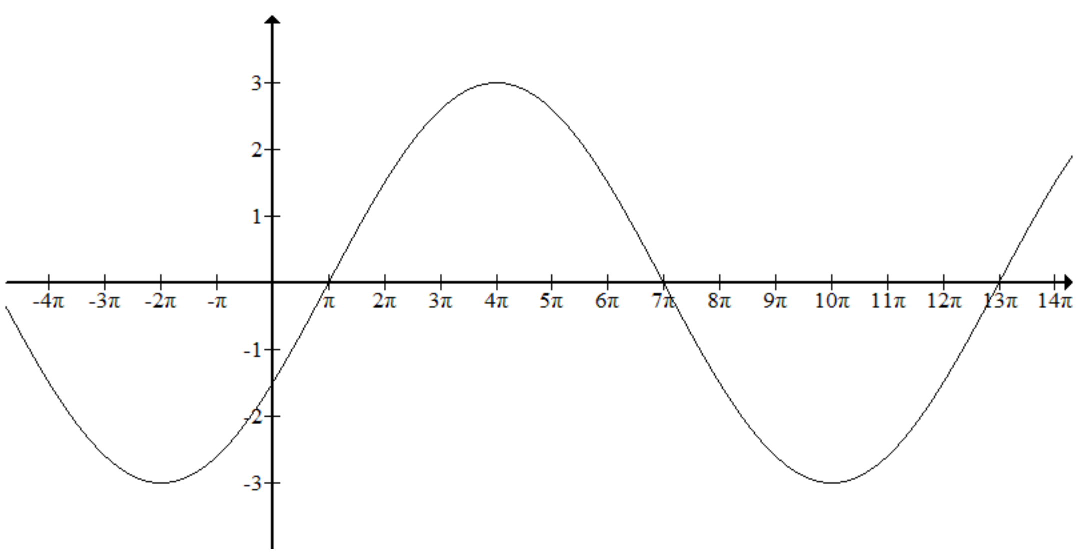
solution: It looks like the amplitude is 2 and the period is 4. Hence

$$B = \frac{2\pi}{4} = \frac{\pi}{2}$$

Since the graph starts out at an extremum, it should be a cos. Cosine graphs usually start at the maximum, but this one starts at a minimum. Thus we need to put a negative in front of the amplitude:

$$y = -2\cos\left(\frac{\pi}{2}t\right)$$

3. Consider



solution: It looks like the amplitude is 3 and the period is 12π, giving us $B = \frac{2\pi}{12\pi} = \frac{1}{6}$.

If it started out at the origin rather than $(\pi, 0)$, the equation would be

$$3 \sin\left(\frac{t}{6}\right)$$

Since we shifted to the right by π, we need to subtract π from x, giving us:

$$3 \sin\left(\frac{t-)}{6}\right)$$

5.4 More trig graphs

The functions tangent and cotangent have period π:

$$\tan(t+\pi) = \tan t \quad \cot(t+\pi) = \cot t$$

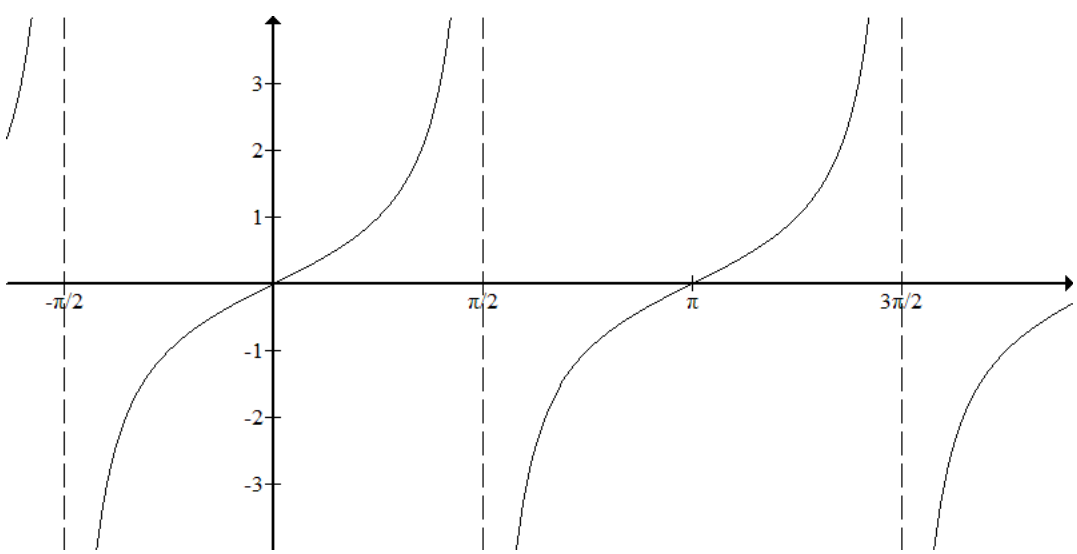
The functions cosecant and secant have period 2π:

$$\csc(t+2\pi) = \csc t \quad \sec(t+2\pi) = \sec t$$

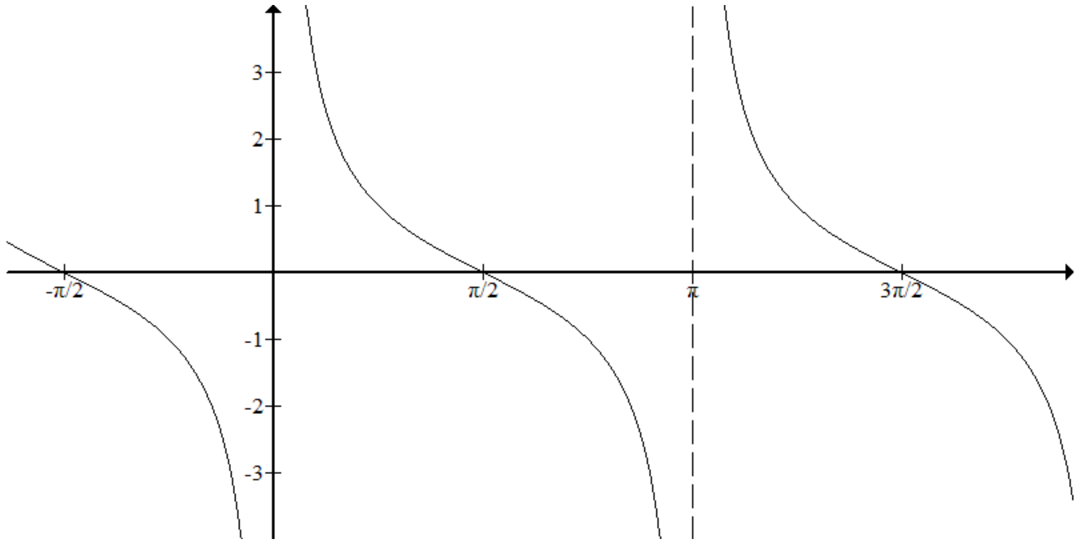
Let's look at what happens to the graph of tangent.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	1.4	1.5	1.55	1.57	1.5707
$\tan t$	0	0.58	1.00	1.75	5.80	14.10	48.08	1255.77	10381.33

As you can see, $\tan t \to \infty$ as $t \to \frac{\pi}{2}$ from the left. Since tan is an odd function, it is symmetric about the origin, so as $t \to -\frac{\pi}{2}$, $\tan t \to -\infty$.



Similarly, $\cot t$ looks like



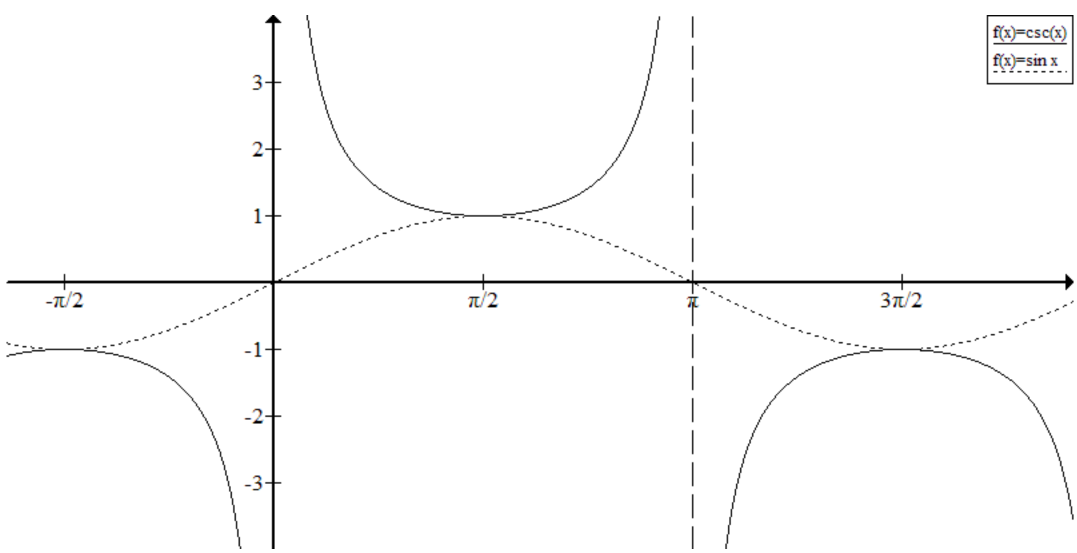
Graphs of secant and cosecant

Recall:

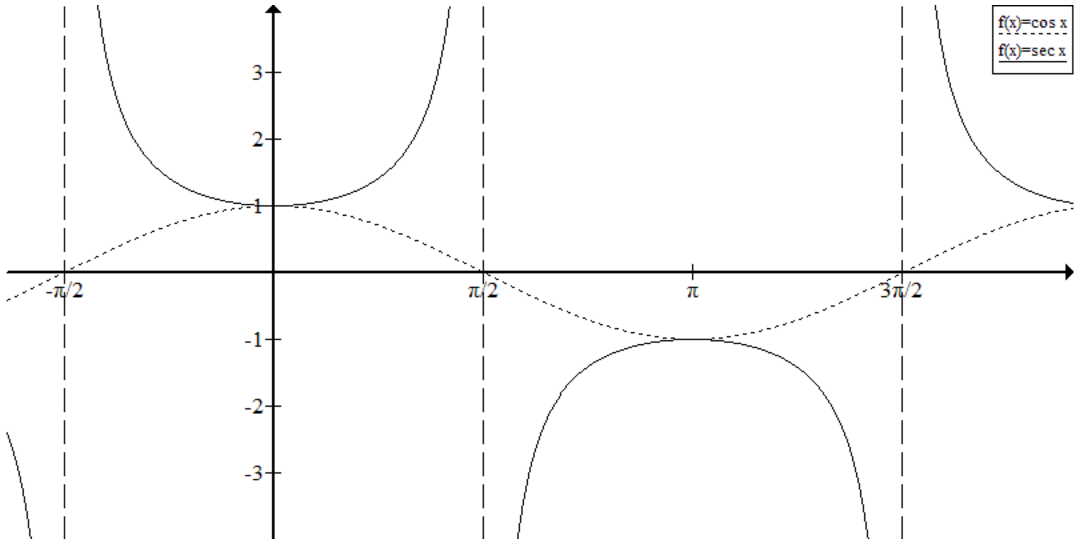
$$\csc(t) = \frac{1}{\sin t} \quad \sec(t) = \frac{1}{\cos t}$$

Since sin and cos both have period 2π, so do sec and csc. Since sin is odd, csc must be odd and thus symmetric about the origin. Since cos is even, sec must be even and thus symmetric about the y-axis.

$\csc(t)$:



$\sec(t)$:

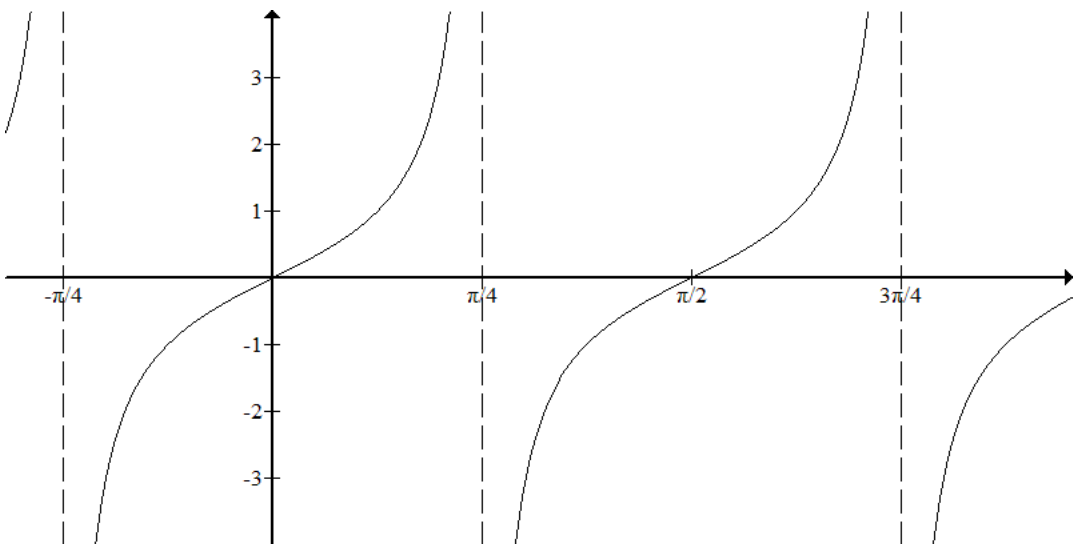


Note: the functions $y = a\tan kx$ and $y = a\cot kx$ have period $\frac{\pi}{k}$.

Examples: Sketch the following graphs:

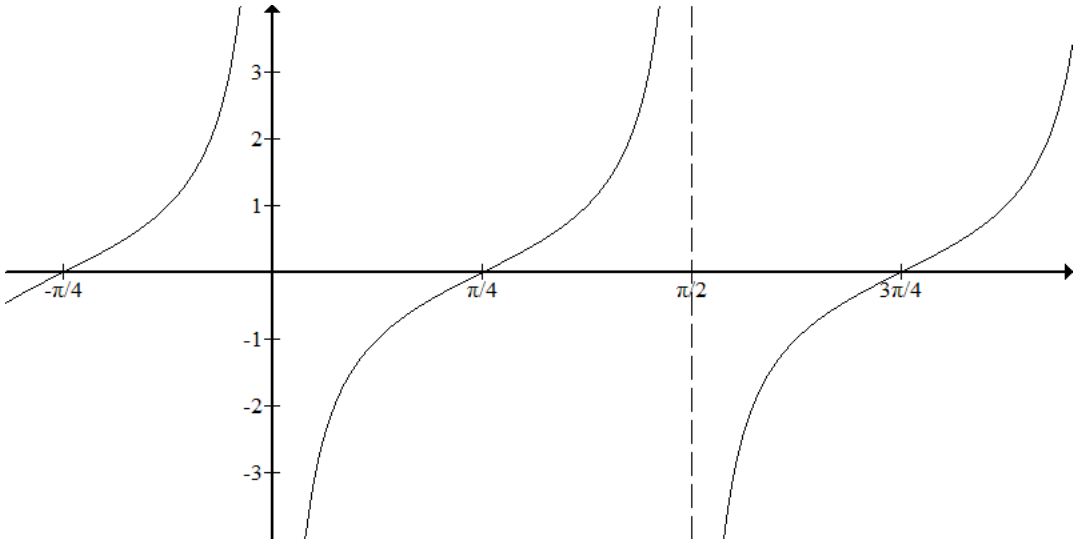
1. $f(x) = \tan(2x)$

   solution: The period here is $\frac{\pi}{2}$ from the formula. Thus we have the following:



2. $g(x) = \tan 2\left(x - \frac{\pi}{4}\right)$.

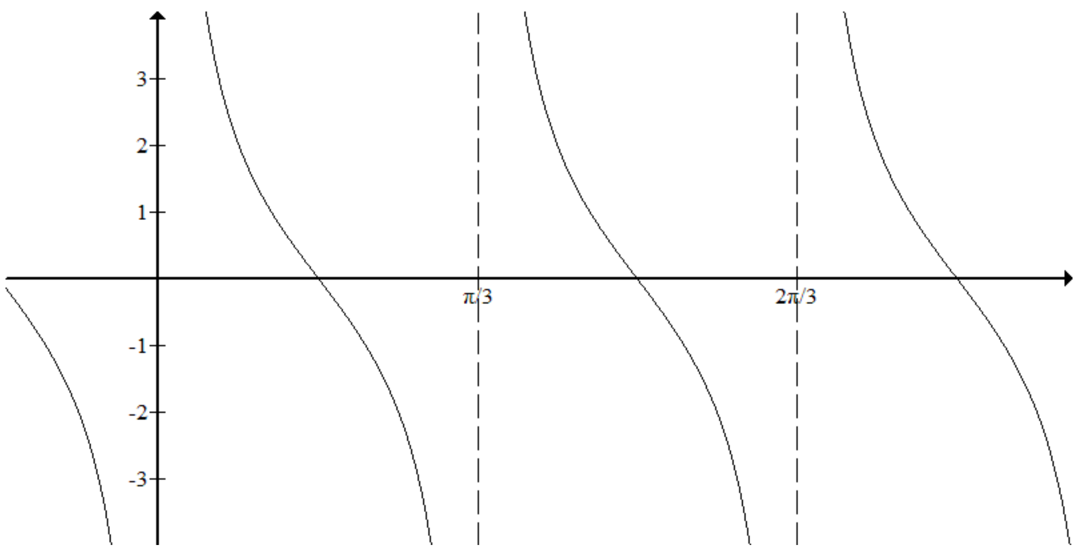
   solution: Once again the period is $\frac{\pi}{2}$, but we have a shift to the right by $\frac{\pi}{4}$ units.



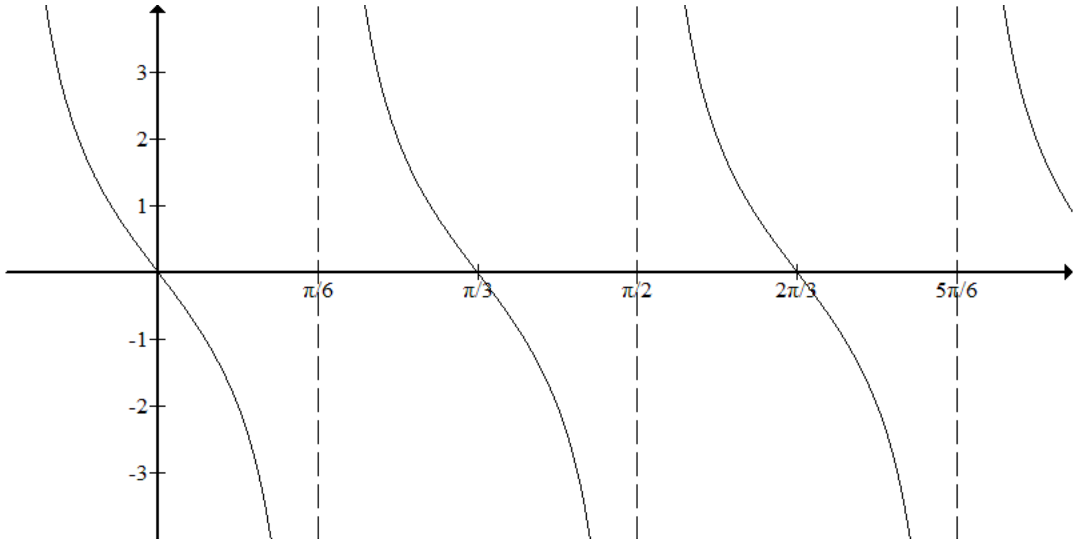
3. $h(x) = 2\cot\left(3x - \frac{\pi}{2}\right)$

<u>*solution:*</u> First note that $h(x) = 2\cot\left(3x - \frac{\pi}{2}\right) = 2\cot 3\left(x - \frac{\pi}{6}\right)$. Thus the period is $\frac{\pi}{3}$ and the function is shifted to the right $\frac{\pi}{6}$.

The graph of $2\cot(3x)$ looks like

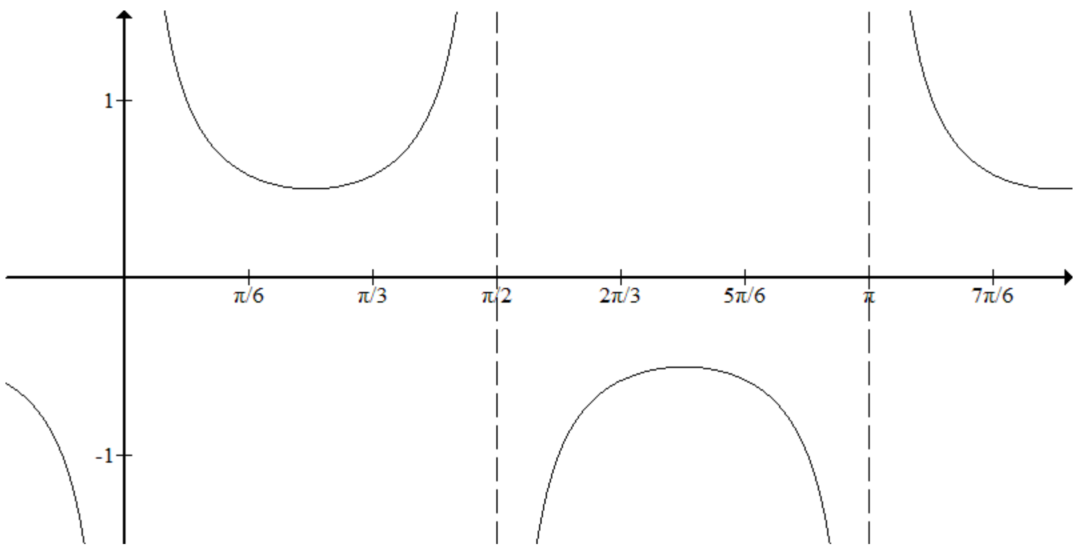


So if we shift to the right by $\frac{\pi}{6}$ units, the asymptotes will be at $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$.



4. $f(x) = \frac{1}{2}\csc(2x)$

   Recall that csc and sec have period $\frac{2\pi}{k}$. Thus the period of our graph is $\frac{2\pi}{2} = \pi$. The $\frac{1}{2}$ out front means the min and max between the asymptotes are at $\frac{1}{2}$ and $-\frac{1}{2}$. Thus we get the following:



5.5 Modelling Harmonic Motion

Periodic behaviour happens a lot in nature. Examples of things that oscillate periodically are daytime temperature, the position of a weight on a spring, and tide level.

If the displacement y of an object (such as a mass on a spring) at time t is

$$y = a \sin \omega t \text{ or } y = a \cos \omega t$$

then the object is said to be in **simple harmonic motion**. In this case,

Amplitude $= |a|$ (maximum displacement)

Period $= \dfrac{2\pi}{\omega}$ (time required to complete one cycle)

Frequency $= \dfrac{\omega}{2\pi}$ (number of cycles per unit time)

If t is measured in seconds, frequency is measured in **Hertz**, Hz.

Example: An object at the end of a spring is set in motion. The displacement from equilibrium (in inches) is found to be

$$y = 10 \sin 4\pi t$$

(a) Find the amplitude, period, and frequency of the motion.

(b) Sketch the graph of the displacement.

solution:

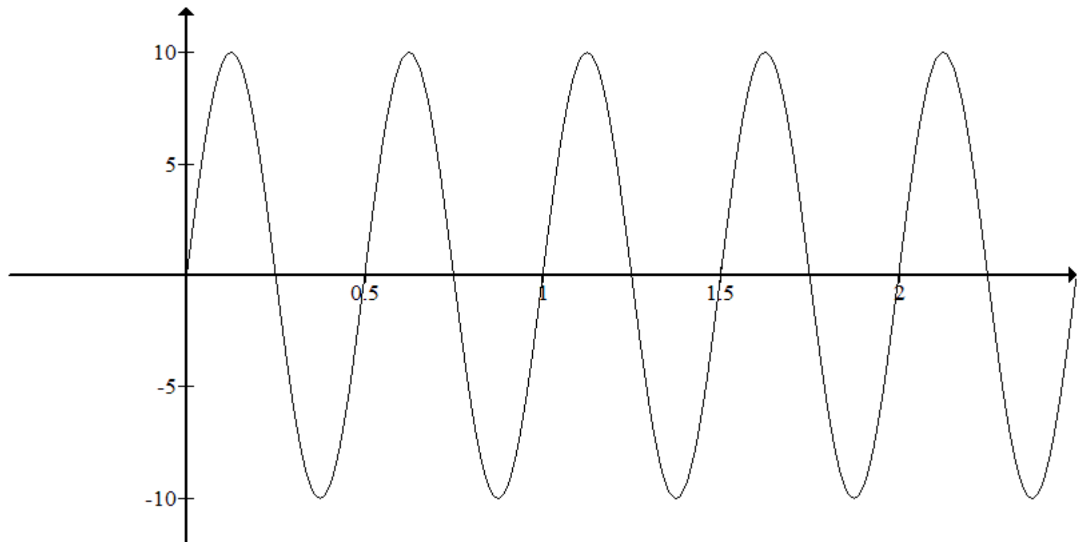
(a) These numbers can be read off the formula:

Amplitude $= |a| = |10| = 10$

Period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{4\pi} = \frac{1}{2}$s

Frequency $= \dfrac{\omega}{2\pi} = \dfrac{4\pi}{2\pi} = 2$Hz

(b) We use our knowledge from the past section to graph the motion:



Example: A mass is suspended from a spring. The spring is compressed a distanceof 4cm and then released at $t = 0$. It is observed that the mass returns to the compressed position after $\frac{1}{3}$s.

(a) Find a function that models the displacement.

(b) Sketch the graph of the displacement.

solution:

(a) The equation will be that of simple harmonic motion. Since the mass returns to a compressed position after $\frac{1}{3}$s, we know that the period is $\frac{1}{3}$s. Thus

$$\begin{aligned} \text{period} &= \frac{2\pi}{\omega} \\ \frac{1}{3} &= \frac{2\pi}{\omega} \\ \omega &= \frac{2\pi}{1/3} \\ \omega &= 6\pi \end{aligned}$$

When $t = 0$, the displacement is 4cm. Thus the function starts at a maximum like cosine, and so we should use the cosine equation. Thus our equation is

$$y = 4\cos 6\pi t$$

(b) Using the rules from the last section, the graph is as follows: