4 Exponential and Logarithmic Functions

4.1 Exponential Functions

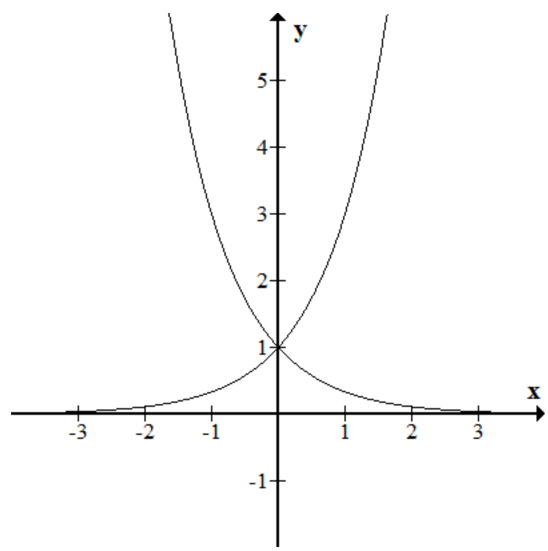
Definition 4.1 *If $a > 0$ and $a \neq 1$, then* **the exponential function with base** a *is given by* $f(x) = a^x$.

Examples: $f(x) = 2^x$, $g(x) = 10^x$, $h(x) = \left(\frac{1}{3}\right)^x$.

Graphs of Exponential Functions

Example: Draw the graph of $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$

x	$f(x)$	$g(x)$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$



Notice that $g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$, so we could have just obtained the graph of g by reflecting f about the y-axis.

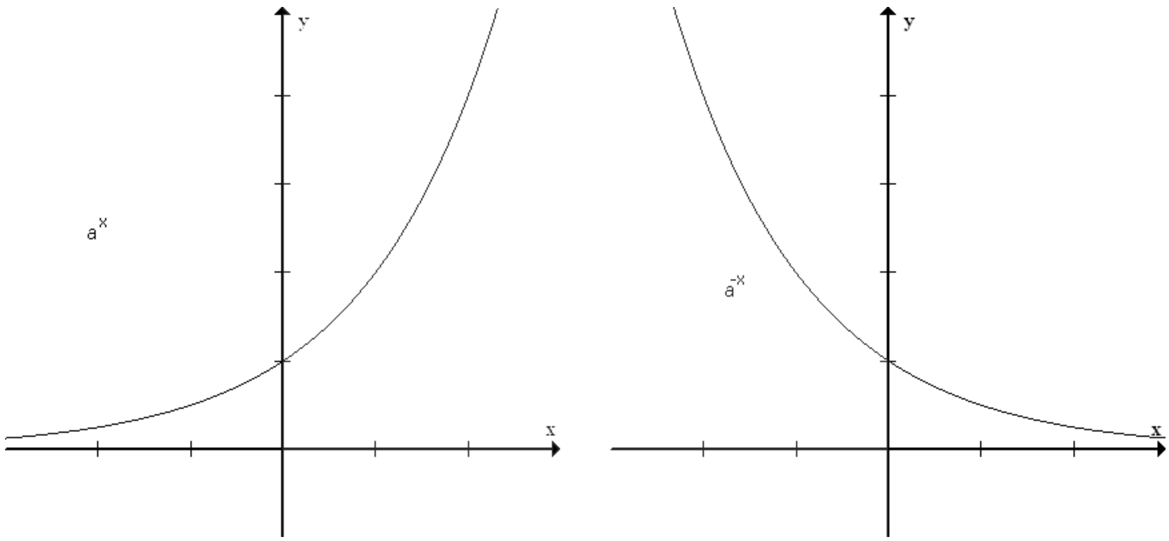
In General:

The exponential function $f(x) = a^x$ with $a > 0, a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$.

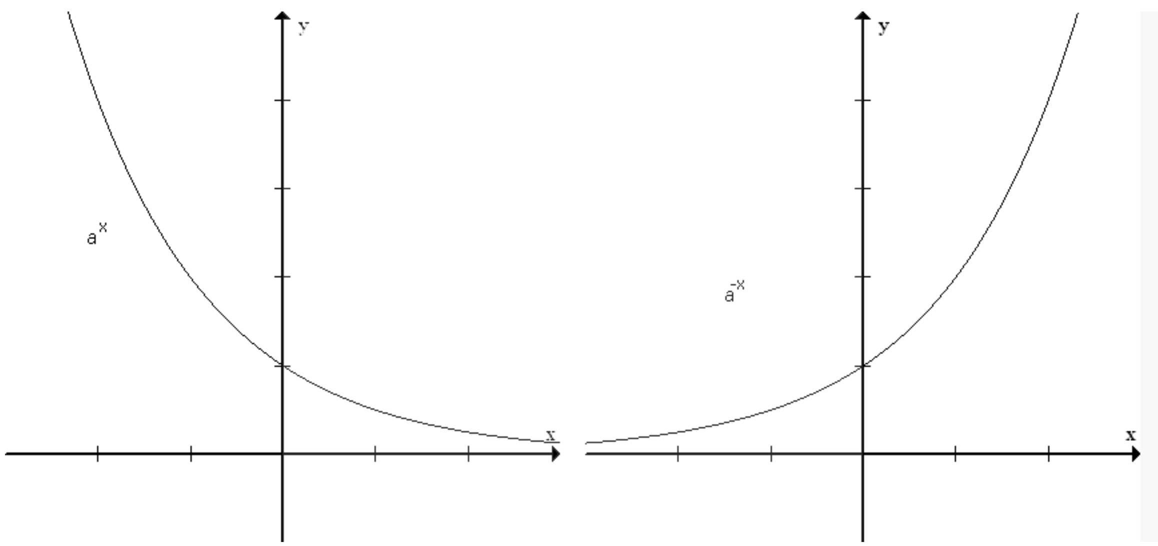
- If $f(x) = a^x$ with $a > 1$, then $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.
- If $f(x) = a^x$ with $0 < a < 1$, then $f(x) \to 0$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

Both have a horizontal asymptote at $y = 0$.

If $a > 0$,



If $a < 1$,

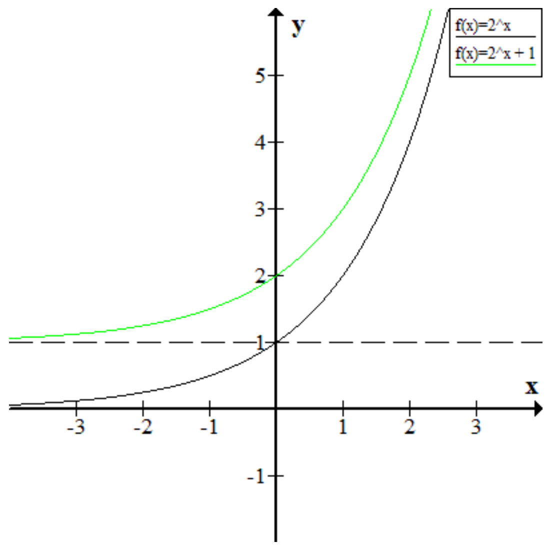


Transformations of Exponential Functions

Examples: Use the graph of $f(x) = 2^x$ to graph the following functions:

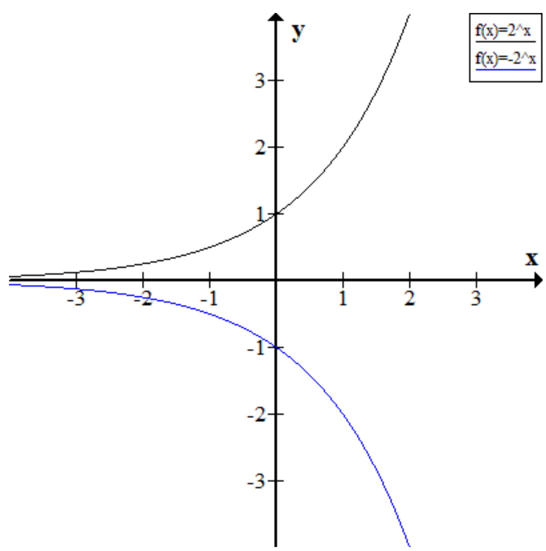
1. $g(x) = 1 + 2^x$

   solution: The graph of g is the graph of f shifted up by 1 unit. This means the horizontal asymptote also shifts up.



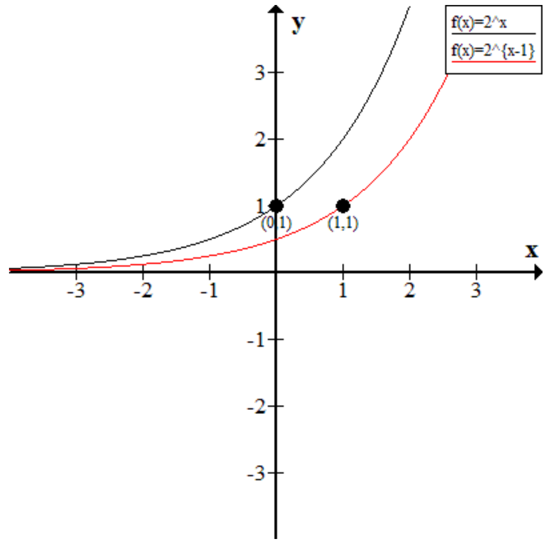
2. $h(x) = -2^x$

   solution: Since we are taking the negative of the whole function, we should reflect the whole graph about the x-axis.



3. $k(x) = 2^{x-1}$

   solution: Since we are subtracting 1 from the x, we should shift the graph to the right by 1.



The most common exponentials used in mathematics are

1. Base 2: $f(x) = 2^x$
2. Base 10: $f(x) = 10^x$
3. Base e: $f(x) = e^x$

The number e is the value that the expression $\left(1 + \frac{1}{n}\right)^n$ approaches as $n \to \infty$.

$$e \approx 2.7182818...$$

Application: Compound Interest

If an amount of money P is invested at an interest rate i per time period, then after one time period, the amount of money A you would have is

$$A = P + P \cdot i = P(1 + i)$$

If you reinvest, then your new sum of money is

$$A = P(1 + i)(1 + i) = P(1 + i)^2$$

Similarly, after 3 time periods you will have $A = P(1 + i)^3$. After k time periods, you have

$$A = P(1 + i)^k$$

Note: this is an exponential function of base $(1 + i)$.

General Formula: If the annual interest rate is r and if interest is compounded n times per year, then the amount of money A you have after t years is given by the **compound interest formula**:

$$\boxed{A(t) = P\left(1 + \frac{r}{n}\right)^{nt}}$$

Where

$A(t) =$ amount after t years,

$P =$ initial deposit (or *principal*),

$r =$ interest rate, expressed as a decimal,

$n =$ number of coumpoundings per year,

$t =$ number of years

Example: a sum of \$ 1000 is invested at an interest rate of 12% per year. Find the amounts in the account after 3 years if the interest is compounded annually, semiannually, quarterly, monthly, and daily.

solution: We have, initially, $P = 1000$. The interest rate is $r = 0.12$, and $t = 3$. Thus we need to calculate $A(t)$ for different values of n.

Compounding	n	Amount after 3 years
Annually	1	$1000\left(1 + \frac{0.12}{1}\right)^{1(3)} = \$1404.93$
Semiannually	2	$1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = \$1418.52$
Quarterly	4	$1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = \$1425.76$
Monthly	12	$1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = \$1430.77$
Daily	365	$1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$

There is another kind of compound interest called **continuous compounding**. The formula for continuous compounding is

$$\boxed{A(t) = Pe^{rt}}$$

Where

$A(t) =$ amount after t years,

$P =$ initial deposit (or *principal*),

$r =$ interest rate, expressed as a decimal,

t = number of years

(For those interested, this expression is what you get if you let $n \to \infty$ in the expression for normal compound interest, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$).

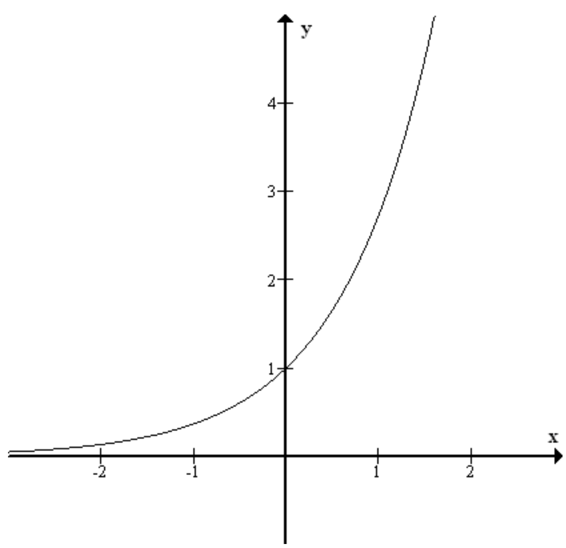
Example: Use the same values from the last example to calculate your total amount with an interest rate compounded continuously.

solution: $A(3) = 1000e^{0.12(3)} = \$1433.33$

In general, continuous compounding provides an "upper bound" for how much return you can get from regular compounding. No matter how many times you compound your interest during the year you will never end up with more money than the formula for coninuous compounding gives you.

4.2 Logarithmic Functions

Note the graph of e^x



This graph passes the horizontal line test, so $f(x) = e^x$ is one-to-one and therefore has an inverse function. This is also true of $f(x) = a^x$ for any $a > 0, a \neq 1$.

Definition 4.2 *Let a be a positive number with $a \neq 1$. The* **logarithmic function with base a**, *denoted by* $\log_a$, *is defined by*

$$\log_a x = y \Leftrightarrow a^y = x$$

The logarithm base a is the inverse function of a^x.

Each exponential equation has a corresponding logarithmic equation

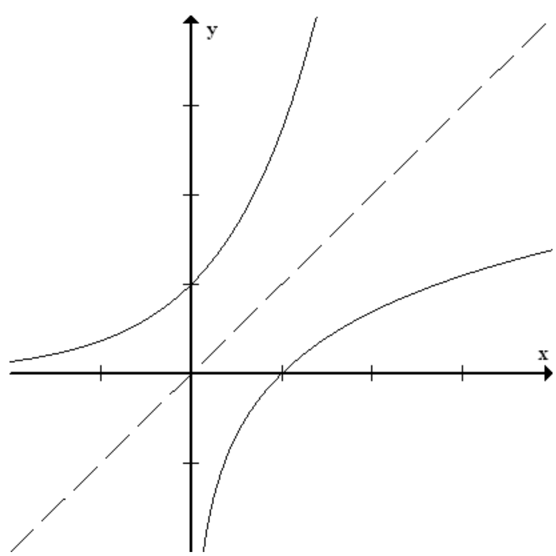
Exponential equation	Logarithmic equation
$10^5 = 10000$	$\log_{10}(10000) = 5$
$2^3 = 8$	$\log_2(8) = 3$
$2^{-3} = \frac{1}{8}$	$\log_2(\frac{1}{8}) = -3$

General Rules:

1. $\log_a 1 = 0$ since $a^0 = 1$
2. $\log_a a = 1$ since $a^1 = a$
3. $\log_a(a^x) = x$ since $a^x = a^x$
4. $a^{\log_a x} = x$

Graphs of Logarithmic Functions

To graph $log_a(x)$, you reflect the graph of a^x about the line $y = x$. From the graph it is clear that **the domain of any logarithm is** $(0, \infty)$.



The logarithm with base e is called the **natual logarithm** and is denoted as ln:

$$\ln(x) = \log_e(x).$$

This is read as either "the natural logarithm of x" or "lawn x"

Properties:

1. $\ln 1 = 0$ since $e^0 = 1$
2. $\ln e = 1$ since $e^1 = e$
3. $\ln(e^x) = x$ since $e^x = e^x$
4. $e^{\ln x} = x$

Examples:

1. $\ln(e^8) = 8$
2. $\ln(\frac{1}{e^2}) = \ln(e^{-2}) = -2$
3. $\ln(5) \approx 1.609$ using the ln key on your calculator.

Domains of Logarithms

As mentioned before, the domain of $\log_a(x)$ is $(0, \infty)$. In other words, taking the log of a negative number or 0 is not allowed.

Examples: Find the domains of the following functions.

1. $f(x) = \ln(e^x - 1)$

   solution: We are only allowed to take the ln of a positive number, therefore we need

$$e^x - 1 > 0$$
$$e^x > 1$$
$$x > 0$$

   So $D_f = (0, \infty)$

2. $g(x) = \ln(4 - x^2)$

   solution: We need what's inside the ln to be positive, so

$$4 - x^2 > 0$$
$$(2 - x)(2 + x) > 0$$

   So we need to do a sign table

x	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
$2+x$	$-$	$+$	$+$
$2-x$	$+$	$+$	$-$
$(2-x)(2+x)$	$-$	$+$	$-$

So $4-x^2>0$ on $(-2,2)$, and thus $D_g=(-2,2)$

3. $h(x)=\ln(\sqrt{x-1})$

   <u>*solution:*</u> We need $\sqrt{x-1}>0$ which is always satisfied unless $x=1$. We also need the expression under the root to be positive,

$$x-1\geq 0$$

$$x\geq 1$$

   So we need $x\geq 1$ and $x\neq 1$. Hence the domain is $D_h=(1,\infty)$.

4.3 Laws of Logarithms

Let a be a positive number with $a\neq 1$. Let x,y and r be any real numbers with $x,y>0$. Then

1. $\log_a(x\cdot y)=\log_a(x)+\log_a(y)$
2. $\log_a\left(\frac{x}{y}\right)=\log_a(x)-\log_a(y)$
3. $\log_a(x^r)=r\log_a(x)$

The proofs of all of these can be found in the book, p 353.

<u>Example:</u> Evaluate each expression.

1. $\log_4 2+\log_4 32$

   <u>*solution:*</u> $\log_4 2+\log_4 32=\log_4(2\cdot 32)=\log_4(64)=3$

2. $\log_2 80-\log_2 5$

   <u>*solution:*</u> $\log_2 80-\log_2 5=\log_2(\frac{80}{5})=\log_2(16)=4$

3. $-\frac{1}{3}\log_1 0(8)$

   <u>*solution:*</u> $-\frac{1}{3}\log_{10}(8)=\log_{10}(8^{-1/3})=\log_{10}(\frac{1}{2})\approx -0.301$

Example: Expand the following expressions:

1. $\log\left(\left(\frac{xy^2}{z}\right)^3\right)$.

   solution:

$$\begin{aligned}\log\left(\left(\frac{xy^2}{z}\right)^3\right) &= 3\log\left(\frac{xy^2}{z}\right)\\ &= 3(\log(xy^2)-\log(z))\\ &= 3(\log x+\log(y^2)-\log z)\\ &= 3(\log x+2\log y-\log z)\end{aligned}$$

2. $\log_2(6x)$.

   solution:

$$\begin{aligned}\log_2(6x) &= \log_2(6)+\log_2(x)\\ &= \log_2(3\cdot 2)+\log_2(x)\\ &= \log_2(3)+\log_2(2)+\log_2(x)\\ &= \log_2(3)+1+\log_2(x)\end{aligned}$$

3. $\ln\left(\frac{ab}{\sqrt{c}}\right)$

   solution:

$$\ln\left(\frac{ab}{\sqrt{c}}\right)=\ln a+\ln b-\frac{1}{2}\ln c$$

Example: Write the following expressions into a single logarithm.

1. $3\log(x)+\frac{1}{2}\log(x+1)$

   solution:

$$3\log(x)+\frac{1}{2}\log(x+1)=\log(x^3)+\log(\sqrt{x+1})=\log(x^3\sqrt{x+1})$$

2. $3\ln 5+\frac{1}{2}\ln x-4\ln(x^2+1)$

   solution:

$$3\ln 5+\frac{1}{2}\ln x-4\ln(x^2+1)=\ln(5^3)+\ln(\sqrt{x})-\ln((x^2+1)^4)=\ln\left(\frac{5^3\sqrt{x}}{(x^2+1)^4}\right)$$

WATCH OUT!

- $\log(x+y) \neq \log x + \log y$
- $\frac{\log(x)}{\log(y)} \neq \log(\frac{x}{y}$
- $(\log(x))^n \neq n\log(x)$

Change of Base

Let $y = \log_b(x)$. Then $b^y = x$. If we take the $\log_a$ of both sides we get

$$\log_a(b^y) = \log_a(x)$$

$$y\log_a(b) = \log_a(x)$$

$$y = \frac{\log_a(x)}{\log_a(b)}$$

Since $y = \log_b(x)$, we have proved the following formula:

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}}$$

Since your calculator does $\log_{10}$ and ln, we can use this formula to calculate a log in any base by taking $a = 10$ or $a = e$.

Example: Evaluate the following logarithms

1. $\log_8(5) = \dfrac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$
2. $\log_9(20) = \dfrac{\ln 20}{\ln 9} \approx 1.36342$

Example: Sketch the graph of $f(x) = \log_6(x)$

solution: $f(x) = \log_6(x) = \frac{\ln x}{\ln 6}$

Since $\frac{1}{\ln 6} \approx 0.55$, we can sketch the graph of $f(x)$ by shrinking the graph of $y = \ln x$ by a factor of 0.55 in the y-direction.

4.4 Exponential and Logarithmic Functions

Examples: Solve the following expressions

1. $\log_{10} x = 2 \Leftrightarrow 10^2 = x \Leftrightarrow x = 100$

2. $\ln(x-2) = 4 \Leftrightarrow x - 2 = e^4 \Leftrightarrow x = e^4 + 2$

   So in order to solve for x we need to get rid of the ln by taking the exponential of both sides.

3. $(\ln x)^2 - \ln(x^2) = -1$.

   solution: Let's start by rewriting it as follows

   $$(\ln x)^2 - 2\ln x + 1$$

   If we let $y = \ln x$, then the equation becomes a quadratic

   $$y^2 - 2y + 1 = 0$$

   $$(y-1)^2 = 0$$

   Thus our solution is $y = 1$. But $y = \ln x$, so

   $$y = \ln x = 1$$

   $$x = e$$

4. $4^x = 9 \Leftrightarrow x = \log_4(9)$

   We could also try taking the ln of both sides

   $$\begin{aligned} 4^x &= 9 \\ \ln(4^x) &= \ln 9 \\ x\ln(4) &= \ln 9 \\ x &= \frac{\ln 9}{\ln 4} \end{aligned}$$

5. $e^{-x} = 2$

   solution:

   $$\begin{aligned} e^{-x} &= 2 \\ \ln(e^{-x}) &= \ln 2 \\ -x\ln(e) &= \ln 2 \\ x &= -\ln 2 \end{aligned}$$

6. $e^x - e^{-x} = 2$

<u>*solution:*</u>

$$\begin{aligned} e^x - e^{-x} &= 2 \\ e^x - \frac{1}{e^x} &= 2 \\ \frac{e^x e^x - 1}{e^x} &= 2 \\ (e^x)^2 - 1 &= 2e^x \\ (e^x)^2 + -2e^x - 1 &= 0 \end{aligned}$$

If we take $y = e^x$, then this is a quadratic again

$$y^2 - 2y - 1 = 0$$

$$y = \frac{2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Since $y = e^x$ we have two solutions:

$$e^x = 1 + \sqrt{2} \text{ and } e^x = 1 - \sqrt{2}$$

Since e^x is always positive, $e^x = 1 - \sqrt{2}$ doesn't make sense, so our only solution is

$$e^x = 1 + \sqrt{2}$$

$$x = \ln(1 + \sqrt{2})$$

7. $e^{3-2x} = 4$

<u>*solution:*</u>

$$3 - 2x = \ln 4$$

$$2x = 3 - \ln 4$$

$$x = \frac{3 - \ln 4}{2}$$

8. $7^t = 4^{t+3}$

<u>*solution:*</u> We start by taking ln of both sides:

$$\ln(7^t) = \ln(4^{t+3})$$

$$t \ln 7 = (t + 3) \ln 4$$

$$t \ln 7 - t \ln 4 = 3 \ln 4$$

$$t(\ln 7 - \ln 4) = 3 \ln 4$$

$$t = \frac{3 \ln 4}{\ln 7 - \ln 4}$$

9. $\log_4(2x+1) = \log_4(x) + \log_4(3)$

   solution:

$$\log_4(2x+1) = \log_4(x) + \log_4(3)$$

$$\log_4(2x+1) = \log_4(3x)$$

$$2x+1 = 3x$$

$$x = 1$$

10. $3xe^x + x^2e^x = 0$

    solution: Factoring, we get

$$xe^x(3+x) = 0$$

    So one of the terms must be 0. Thus, either $e^x = 0$, $x = 0$ or $(3+x) = 0$. e^x is never 0, so the only solutions are $x = 0$ and $x = -3$.

11. $\log_{10}(x+2) + \log_{10}(x-1) = 1$

    solution:

$$\begin{aligned}
log_{10}(x+2) + \log_{10}(x-1) &= 1 \\
\log_{10}((x+2)(x-1)) &= 1 \\
(x+2)(x-1) &= 10 \\
x^2 + x - 2 - 10 &= 0 \\
x^2 + x - 12 &= 0 \\
(x+4)(x-3) &= 0
\end{aligned}$$

    Thus the solutions are $x = -4$ and $x = 3$. However, $x = -4$ causes the original expression not make sense, so our only solution is $x = 3$.

Example: A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded

a) Semiannually,

b) Continuously.

solution:

a) P = initial deposit = 5000

   r = interest rate = 0.05

   n = times compounded = 2

   $A(t)$ = end total = 10000

$$\begin{aligned} A(t) &= P\left(1+\frac{r}{n}\right)^{nt} \\ 10000 &= 5000\left(1+\frac{0.05}{2}\right)^{2t} \\ 2 &= (1.025)^{2t} \\ \ln 2 &= \ln(1.025^{2t}) \\ \ln 2 &= 2t\ln(1.025) \\ t &= \frac{\ln 2}{2\ln(1.025)} \approx 14.03 \end{aligned}$$

So compounded semiannually it takes about 14 years to double.

b) Here the numbers are the same but we use the continuous compounding formula $A(t) = Pe^{rt}$

$$\begin{aligned} A(t) &= Pe^{rt} \\ 10000 &= 5000e^{0.05t} \\ 2 &= e^{0.05t} \\ \ln 2 &= 0.05t \\ t &= \frac{\ln 2}{0.05} \approx 13.86 \end{aligned}$$

So it takes about 13.86 years to double if compounded continuously.

4.5 Modelling with Exponential and Logarithmic Functions

Exponential Models of Population Growth

A population that experiences **exponential growth** increases according to the equation

$$n(t) = n_0 e^{rt}$$

Where

$n(t)$ = population at time t,

n_0 = initial population,

r = relative rate of growth,

t = time.

Note: This is the same equation as interest compounded continuously, where r is now the rate at which the population grows rather than the interest rate.

Example: The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the relative growth rate is 40% per hour.

a) Find a function that models the number of bacteria after t hours.

b) What is the estimated count after 10 hours?

c) How long does it take the population to triple?

solution:

a) Here the initial population is $n_0 = 500$ and the growth rate is $r = 0.40$. Thus we have

$$n(t) = 500e^{0.4t}$$

b) To find this we just sub in $t = 10$ in the equation:

$$n(10) = 500e^{0.4\cdot 10} \approx 27300$$

c) Here, the final population is $n(t) = 1500$, so we need to solve

$$\begin{aligned} n(t) &= n_0 e^{rt} \\ 1500 &= 500e^{0.40t} \\ 3 &= e^{0.40t} \\ \ln 3 &= 0.40t \\ t &= \frac{\ln 3}{0.40} \approx 2.75 \end{aligned}$$

So it takes the population about 2.75 hours to triple.

Radioactive Decay

A quantity that experiences **exponential decay** decreases according to the model

$$m(t) = m_0 e^{-rt}$$

Where

$m(t) =$ amount at time t,

$m_0 =$ initial population,

$r =$ relative rate of decay,

$t =$ time.

A common way to express the rate of decay is **half-life** - the time required for half the quantity to decay. If h is the half-life, then

$$r = \frac{\ln 2}{h}$$

How do we get this? Suppose we start with $m_0 = 1$ and end up with $m(t) = \frac{1}{2}$. The time it takes to do this is the half-life h, thus

$$m(h) = \frac{1}{2} = e^{-rh}$$

$$\ln \frac{1}{2} = -rh$$

$$r = -\frac{\ln \frac{1}{2}}{h} = \frac{\ln 2}{h}$$

The last step is due to the fact that $\ln \frac{1}{2} = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$.

Example: Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300mg.

a) Find a function that models the amount of sample left after t days.

b) Find the mass remaining after 1 year.

c) How long will it take for the sample to decay to a mass of 200mg?

solution:

a) We are given $m_0 = 300$ and $h = 140$. Hence

$$r = \frac{\ln 2}{140} \approx 0.00495$$

Thus

$$m(t) = 300e^{-0.00495t}$$

b) 1 year corresponds to a time of $t = 365$, so

$$m(365) = 300e^{-0.00495(365)} \approx 49.26$$

So after a year about 49.26mg are left.

c) Here we set $m(t) = 200$ and solve for t:

$$\begin{aligned}
m(t) &= m_0 e^{rt} \\
200 &= 300e^{-0.00495t} \\
\frac{3}{2} &= e^{-0.00495t} \\
\ln\frac{3}{2} &= -0.00495t \\
t &= -\frac{\ln 32}{0.00495} \approx 81.9
\end{aligned}$$

So after about 82 days we are left with 200mg.