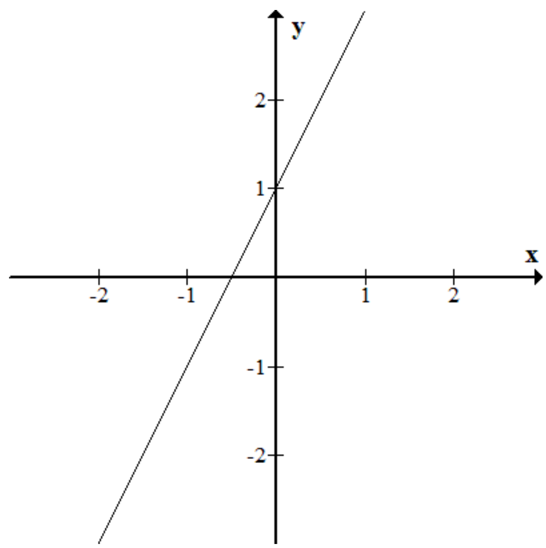
3 Polynomial and Rational Functions

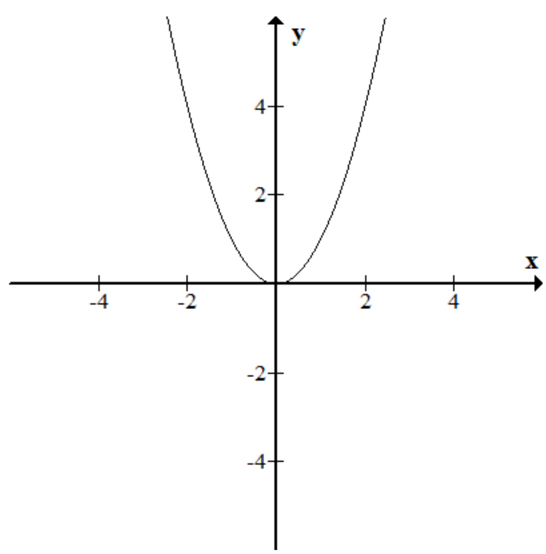
3.1 Polynomial Functions and their Graphs

So far, we have learned how to graph polynomials of degree 0, 1, and 2. Degree 0 polynomial functions are things like $f(x) = 2$, which is a straight horizontal line with constant y coordinate 2.

An example of a degree 1 polynomial is $f(x) = 2x+1$ whose graph is a straight line through $(0, 1)$ with slope 2.

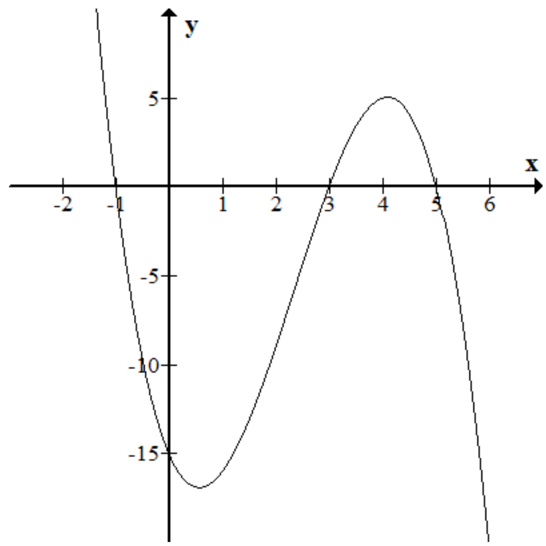


In section 2.7 we went into detail as to what the graphs of degree 2 polynomials look like - they are paraboloas opening either up or down. For example the graph of $f(x) = x^2$ looks like



Graphs of polynomials of higher degrees are more difficult.

Example: Let $f(x) = -(x+1)(x-3)(x-5)$. This function is already factored, and it has x-intercepts $x = 1, 3, 5$.



Notice that as $x \to -\infty$, $f(x) \to \infty$ and that as $x \to \infty$, $f(x) \to -\infty$

In general:

Suppose $f(x)$ is a polynomial.

- If $f(x)$ has even degree and the leading coefficient is positive, $f(x) \to \infty$ as $x \to \pm\infty$.
- If $f(x)$ has even degree and the leading coefficient is negative, $f(x) \to -\infty$ as $x \to \pm\infty$.
- If $f(x)$ has odd degree and the leading coefficient is positive, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.
- If $f(x)$ has odd degree and the leading coefficient is negative, $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

3.2 Dividing Polynomials

Long Division of Polynomials

This process is used to factor higher degree polynomials.

Example: Divide $x^3 - x^2 - 4x + 4$ by $x - 1$.

solution:

$$
\begin{array}{r}
x^2 \qquad\quad -4 \\
x-1\overline{\big)\; x^3 - x^2 - 4x + 4} \\
\underline{-x^3 + x^2 \qquad\qquad} \\
-4x + 4 \\
\underline{4x - 4} \\
0
\end{array}
$$

So therefore $x^3 - x^2 - 4x + 4 = (x-1)(x^2-4) = (x-1)(x-2)(x+2)$, and thus the roots of $x^3 - x^2 - 4x + 4$ are $x = 1, 2, -2$.

What values of x make our function positive?

x	$(-\infty, -2)$	$(-2, 1)$	$(1, 2)$	$(3, \infty)$
$x+2$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$(x-1)(x-2)(x+2)$	$-$	$+$	$-$	$+$

So $x^3 - x^2 - 4x + 4 > 0$ on $(-2, 1) \cup (3, \infty)$.

<u>Note:</u> Here $(x-1)$ being a factor of $f(x) = x^3 - x^2 - 4x + 4$ means that $f(1) = 0$. This, in fact, works both ways – if $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

<u>Example:</u> Let $f(x) = x^3 + 3x^2 + 3x + 1$. Notice that $f(-1) = 0$. Factor $f(x)$ and find the roots of $f(x)$.

solution: Since we are given that $f(-1) = 0$, we know that $(x - (-1)) = x + 1$ must be a factor of $f(x)$. So let's use long division to factor

$$
\begin{array}{r}
x^2 + 2x + 1 \\
x+1\overline{\big)\; x^3 + 3x^2 + 3x + 1} \\
\underline{-x^3 - x^2 \qquad\qquad\quad} \\
2x^2 + 3x \qquad \\
\underline{-2x^2 - 2x \qquad} \\
x + 1 \\
\underline{-x - 1} \\
0
\end{array}
$$

Thus, $x^3 + 3x^2 + 3x + 1 = (x+1)(x^2 + 2x + 1) = (x+1)(x+1)^2 = (x+1)^3$. So $f(x)$ has only one root, $x = -1$.

Here, you were given one of the roots. Sometimes you have to find one yourself. Here is a good general rule: **An integer can only be the root of a polynomial with leading coefficient 1 if it divides the constant term**. So if you are searching for roots of a polynomial with leading coefficient 1, try plugging in values of x that divide the constant term into the equation and see if you get 0.

<u>Example:</u> Let $f(x) = x^3 - x^2 - x - 2$. Find the roots of f and then find where f is negative.

<u>*solution:*</u> We should try some x values that divide 2 to see if they make $f(x) = 0$.

$$f(1) = 1^3 - 1^2 - 1 - 2 = -3 \neq 0$$

$$f(-1) = (-1)^3 - (-1)^2 - (-1) - 2 = -1 \neq 0$$

$$f(2) = (2)^3 - (2)^2 - 2 - 2 = 0$$

Since $f(2) = 0$, we must have that $x - 2$ divides $f(x)$. Using long division:

$$\begin{array}{r|rrrr}
 & x^2 & +x & +1 & \\ \cline{2-5}
x-2 & x^3 & -x^2 & -x & -2 \\
 & -x^3 & +2x^2 & & \\ \cline{2-3}
 & & x^2 & -x & \\
 & & -x^2 & +2x & \\ \cline{3-4}
 & & & x & -2 \\
 & & & -x & +2 \\ \cline{4-5}
 & & & & 0
\end{array}$$

So $f(x) = x^3 - x^2 - x - 2 = (x-2)(x^2 + x + 1)$. On quick inspection it doesn't appear that the quadratic factors. Using the quadratic formula to find its roots:

$$\frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2}$$

Our discriminant is negative, so $x^2 + x + 1$ has no roots. Thus this is as far as we can factor.

x	$(-\infty, 2)$	$(2, \infty)$
$x + 2$	$-$	$+$
$x - 1$	$+$	$+$
$(x-2)(x^2+x+1)$	$-$	$+$

So $f(x) < 0$ on $(-\infty, 2)$.

Synthetic Division

Suppose we divide $p(x) = 2x^3 - 7x^2 + 5$ by $x - 3$. We get

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & 2x^2 & -x & -3 \\
\cline{2-5}
x-3 & 2x^3 & -7x^2 & & +5 \\
 & -2x^3 & +6x^2 & & \\
\cline{2-3}
 & & -x^2 & & \\
 & & x^2 & -3x & \\
\cline{3-4}
 & & & -3x & +5 \\
 & & & 3x & -9 \\
\cline{4-5}
 & & & & -4
\end{array}
$$

We can represent this in shorthand like this

$$
\begin{array}{r|rrrr}
 & 2 & -7 & 0 & 5 \\
3 & & & & \\
\hline
\end{array}
$$

$$
\begin{array}{r|rrrr}
 & 2 & -7 & 0 & 5 \\
3 & & & & \\
\hline
 & 2 & & &
\end{array}
$$

$$
\begin{array}{r|rrrr}
 & 2 & -7 & 0 & 5 \\
3 & & 6 & & \\
\hline
 & 2 & & &
\end{array}
$$

$$
\begin{array}{r|rrrr}
 & 2 & -7 & 0 & 5 \\
3 & & 6 & & \\
\hline
 & 2 & -1 & &
\end{array}
$$

$$
\begin{array}{r|rrrr}
 & 2 & -7 & 0 & 5 \\
3 & & 6 & -3 & \\
\hline
 & 2 & -1 & &
\end{array}
$$

$$\begin{array}{r|rrrr} & 2 & -7 & 0 & 5 \\ 3 & & 6 & -3 & \\ \hline & 2 & -1 & -3 & \end{array}$$

$$\begin{array}{r|rrrr} & 2 & -7 & 0 & 5 \\ 3 & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & \end{array}$$

$$\begin{array}{r|rrrr} & 2 & -7 & 0 & 5 \\ 3 & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

This is called **synthetic division**. We use a 3 instead of a -3 so that we can add instead of subtract. This tells us that $2x^3 - 7x^2 + 5 = (x-3)(2x^2 - x - 3) - 4$. Note that since there is a remainder, $x = 3$ is not a root.

Theorem 3.1 The Remainder Theorem: *If the polynomial $p(x)$ is divided by $x - c$, then the remainder is the value $p(c)$.*

Example: Let $p(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$. Find the quotient and remainder when $p(x)$ is divided by $x + 2$.

solution:

$$\begin{array}{r|rrrrrr} & 3 & 5 & -4 & 0 & 7 & 3 \\ -2 & & -6 & 2 & 4 & -8 & 2 \\ \hline & 3 & -1 & -2 & 4 & -1 & 5 \end{array}$$

So the remainder is 5, hence

$$p(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3 = (x+2)(-6x^4 + 2x^3 + 4x^2 - 8x + 2) + 5.$$

Also, $p(-2) = 5$.

Example: Factor $p(x) = x^3 - 7x + 6$ and find the roots of $p(x)$.

solution: Since the leading coefficient is 1, we should try numbers that divide into 6:

$$f(1) = 1^3 - 7(1) + 6$$

What luck! A hit on the first try. Thus we should divide the polynomial by $x-1$:

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & 6 \\ 1 & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

Note that we are relieved to get 0 as a remainder - that confirms that 1 is a root and that we didn't make a mistake when we found it. This gives us

$$p(x) = x^3 - 7x + 6 = (x-1)(x^2 + x - 6)$$

We can factor the quadratic as normal, $x^2 + x - 6 = (x-2)(x+3)$. Hence

$$p(x) = x^3 - 7x + 6 = (x-1)(x-2)(x+3)$$

Hence the roots are $x = 1, 2, -3$.

3.3 Rational Functions

A **rational function** is a function of the form $r(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomials.

- The domain of a rational function is all real numbers except the ones that make the denominator 0.
- The roots of a rational function are the zeroes of the numerator.

Examples: Find the domain, roots, and y-intercept of the following functions.

1. $f(x) = \frac{7}{x+2}$

   solution: The domain is

   $$D_f = \{x \in \mathbb{R} \mid x \neq 2\}$$

   Since the numerator can never be 0, there are no roots.

   Setting $x = 0$ gives us that the y-intercept is

   $$y = \frac{7}{0+2} = \frac{7}{2}$$

2. $g(x) = \dfrac{x}{x^2 - 3x - 4}$

   solution: The domain is $x \in \mathbb{R}$ except for when $x^2 - 3x + 4 = (x-4)(x+1) = 0$. Thus $x = 4$ and $x = -1$ is not allowed, and so our domain is

   $$D_g = \{x \in \mathbb{R} \mid x \neq 4, -1\}$$

The roots of g are when the numerator equals 0, that is, $x = 0$. The y-intercept is when $x = 0$, or

$$y = g(0) = \frac{0}{0 - 0 - 4} = 0.$$

So the y-intercept is 0.

3. $h(x) = \dfrac{2x^2 + x - 3}{x^2 - 4}$

<u>*solution:*</u> Since the denominator factors to $(x-2)(x+2)$, the domain is all real numbers except $x = 2$ or -2:

$$D_h = \{x \in \mathbb{R} \mid x \neq 2, -2\}$$

To get the roots, we should apply the quadratic formula to the top since there is no obvious factoring:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-3)}}{2(2)} = \frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}$$

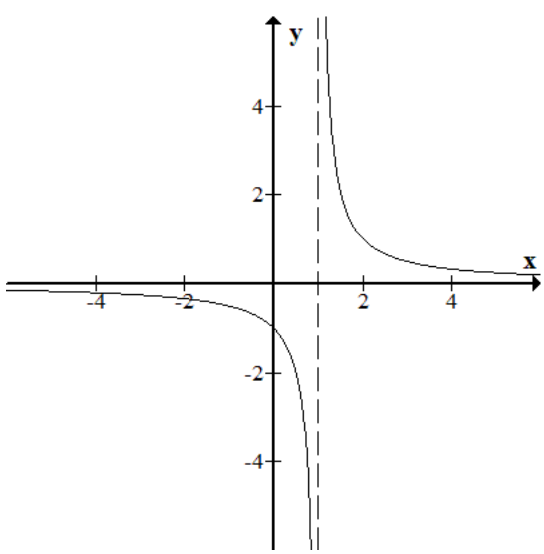
Thus $x = 1$ and $x = -\frac{3}{2}$ are the roots. Note that this also means that $2x^2 + x - 3 = 2(x - 1)(x + \frac{3}{2})$.

The y-intercept is

$$y = h(0) = \frac{2(0)^2 + 0 - 3}{0^2 - 4} = -\frac{3}{4}$$

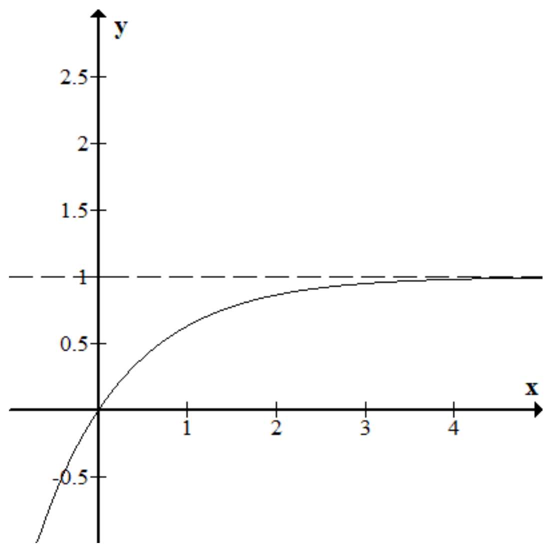
<u>Vertical and Horizontal Asymptotes</u>

Definition 3.1 *The line $x = a$ is a* **vertical asymptote** *of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.*



This graph has a vertical asymptote at $x = 1$.

Definition 3.2 *The line $y = b$ is a* **horizontal asymptote** *of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.*



This graph has a horizontal asymptote at $x = 1$.

Example: Let $f(x) = \dfrac{1}{x}$.

The domain of $f(x)$ is $D_f = \{x \in \mathbb{R} \| \ x \neq 0\}$.

Let's look at how f behaves near 0.

x	$f(x)$
-0.1	-10
-0.01	-100
-0.001	-1000

x	$f(x)$
0.1	10
0.01	100
0.001	1000

As the x values get closer and closer to 0 from the negative side, f's values get closer and closer to $-\infty$. On the other hand, as x values get closer and closer to 0 from the positive side, f's values get larger and larger. This means we have a vertical asymptote at $x = 0$.

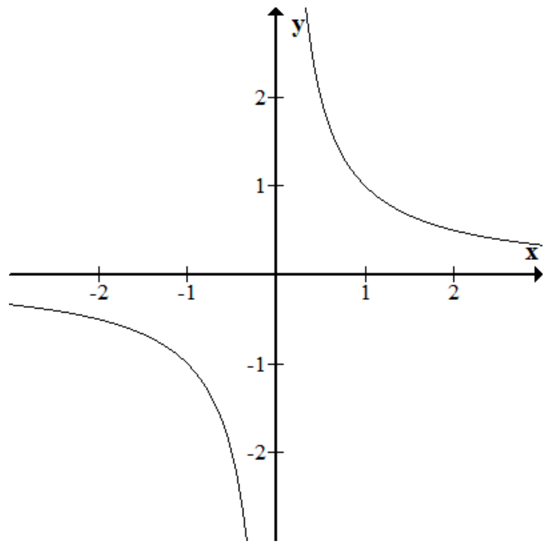
Now let us look at what f does as x gets very large in both directions.

x	$f(x)$
-10	-0.1
-100	-0.01
-1000	-0.001

x	$f(x)$
10	0.1
100	0.01
1000	0.001

So as x goes to either $+\infty$ or $-\infty$, the values of f approach 0. This means we have a horizontal azymptote at $x = 0$.

To sketch the rest we can do a table of values. What we end up with is



This graph never crosses either of the axes but gets close to both of them.

Transformations of $\frac{1}{x}$

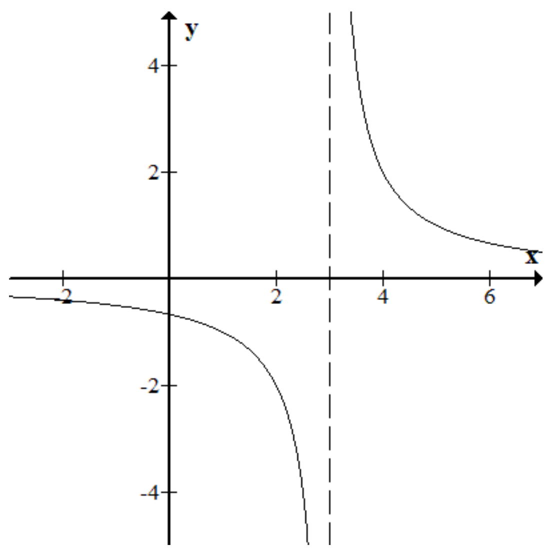
Any rational function of the form $r(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$ can be graphed by shifting, stretching, and/or reflecting the graph of $f(x) = \dfrac{1}{x}$. The way we do this is by polynomial division.

Examples: Let $f(x) = \dfrac{1}{x}$

1. Sketch $g(x) = \dfrac{2}{x-3}$.

   solution: $g(x) = \dfrac{2}{x-3} = 2\left(\dfrac{1}{x-3}\right) = 2f(x-3)$.

   According to our rules about transforming functions, we can obtain the graph of g by shifting the graph of $f(x) = \frac{1}{x}$ by 3 units to the right, and stretching vertically by a factor of 2. Here, $g(x)$ has a vertical asymptote at $x = 3$ and a horizontal asymptote at $y = 0$.



2. Sketch $h(x) = \dfrac{3x+5}{x+2}$

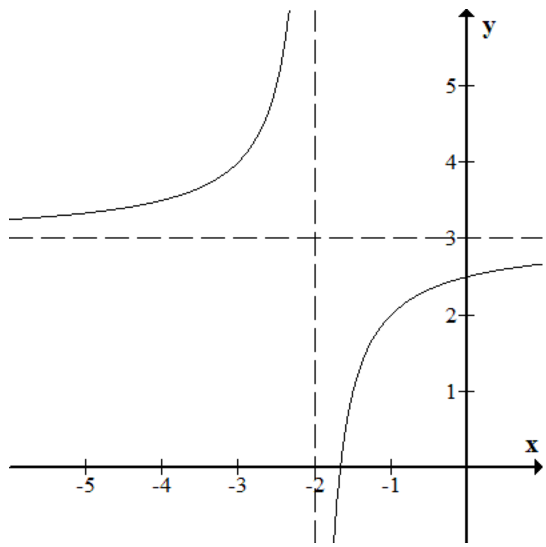
   solution: Here we start by dividing the two polynomials

$$\begin{array}{r|r} & 3 \\ \hline x+2 & 3x+5 \\ & -3x-6 \\ \hline & -1 \end{array}$$

   The remainder is -1, so

$$\frac{3x+5}{x+2} = 3 + \frac{-1}{x+2} = -f(x+2) + 3$$

According to our rules about transforming functions, we can obtain the graph of h by shifting the graph of $f(x) = \frac{1}{x}$ by 2 units to the left, shifting 3 units up, and reflecting in the x-axis. Here, $h(x)$ has a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 3$.



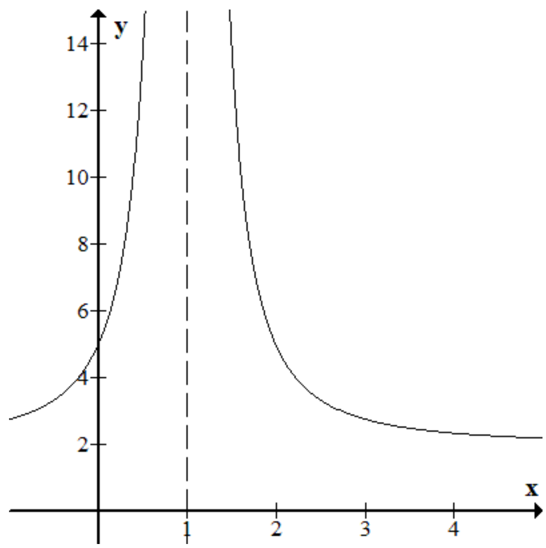
3. Sketch $r(x) = \dfrac{2x^2 - 4x + 5}{x^2 - 2x + 1}$.

<u>*solution:*</u> We have to find a few things first.

- Domain – the denominator factors into $(x-1)^2$, so the only x value that is not allowed is $x = 1$. Thus
$$D_r = \{x \in \mathbb{R} \mid x \neq 1\}$$
- Vertical asymptotes – these occur when the denominator is 0. Thus we have a vertical asymptote at $x = 1$.
- Horizontal asymptote – to find this we divide each term by the highest exponent in the denominator and look at when $x \to \infty$.
$$r(x) = \frac{2x^2 - 4x + 5}{x^2 - 2x + 1} = \frac{\frac{2x^2}{x^2} - \frac{4x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{1}{x^2}} = \frac{2 - \frac{4}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$
As $x \to \infty$, all the terms in that quotient disappear except for the 2 on the top and the 1 on the bottom. Hence, as $x \to \infty$, $r(x) \to \frac{2}{1} = 2$.
- Behaviour near asymptotes – now we have to look at what is happening to our function near our asymptotes.

x	y
0	5
6.5	14
0.9	302
0.99	50,002

x	y
2	5
1.5	14
1.1	302
1.01	30,002



In general, let $r(x)$ be a rational function

1. The vertical asymptotes of $r(x)$ are the roots of the denominator.
2. The horizontal asymptotes are determined as follows:
   - If the degree of the top is larger than the degree of the bottom, there are no horizontal asymptotes.
   - If the degree of the top is smaller than the degree of the bottom, there is a horizontal asymptote at $y = 0$.
   - If the degrees of the top and bottom are the same, then there is a horizontal asymptote at $y = \frac{a}{b}$, where a is the leading coefficient of the top and b is the leading coefficient of the bottom.

Example: $\dfrac{3x^2 - 2x - 1}{2x^2 + 3x - 2}$

- Vertical asymptotes – we use the quadratic equation to find the roots of the denominator

$$x = \frac{-3 \pm \sqrt{9 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm 5}{4} = -2, \frac{1}{2}$$

  So $x = -2$ and $x = \frac{1}{2}$ are the vertical asymptotes.

- Horizontal asymptotes – the degrees are the same, so the horizontal asymptote occurs at the ratio of the leading coefficients, in this case the leading term on the top is $3x^2$ while that on the bottom is $2x^2$, so the ratio of the coefficients is $\dfrac{3}{2}$. Thus the horizontal asymptote is at $y = \dfrac{3}{2}$.

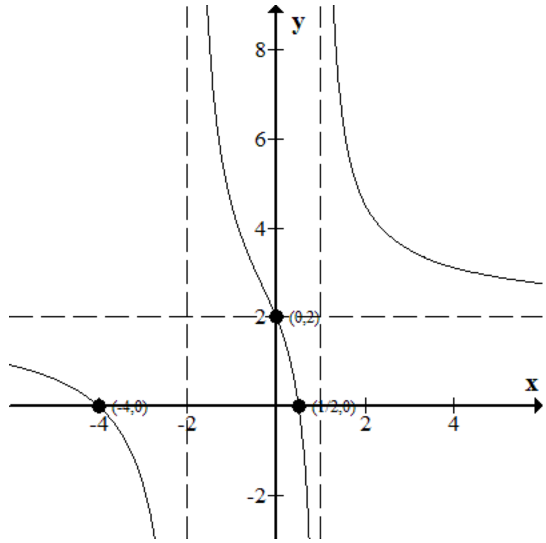
Sketching Graphs of Rational Functions

- Find the domain of f by factoring the denominator.
- Factor the numerator.
- Find the x- and y-intercepts.
- Find the vertical asymptotes.
- Find the horizontal asymptotes.
- Analyze the behaviour of f around the asymptotes.
- Sketch the graph.

Example: Sketch $r(x) = \dfrac{2x^2 + 7x - 4}{x^2 + x - 2}$

- $x^2 + x - 2 = (x-1)(x+2)$. So $D_r = \{x \in \mathbb{R} \mid x \neq 1, -2\}$
- $$x = \frac{-7 \pm \sqrt{49 - 4(2)(-4)}}{2(2)} = \frac{-7 \pm 9}{4} = -4, \frac{1}{2}$$
  This means that $2x^2 + 7x - 4 = 2(x - \frac{1}{2})(x+4) = (2x-1)(x+4)$. Thus so far we have
  $$r(x) = \frac{(2x-1)(x+4)}{(x-1)(x+2)}$$
- The x-intercepts are the roots of the numerator. In this case, the roots are $x = -4$ and $x = \frac{1}{2}$.
  The y-intercept we get by subbing in $x = 0$:
  $$r(x) = \frac{2(0)^2 + 7(0) - 4}{0^2 + 0 - 2} = 2$$
  So the y-intercept is $y = 2$.
- We have vertical asymptotes where the denominator is 0, ie, at $x = 1$ and $x = -2$.

$x \to$	-2 from the left	-2 from the right	1 from the left	1 from the right
$r(x) = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)}$	$\dfrac{(-)(+)}{(-)(-)}$	$\dfrac{(-)(+)}{(-)(+)}$	$\dfrac{(+)(+)}{(-)(+)}$	$\dfrac{(+)(+)}{(+)(+)}$
$y \to$	$-\infty$	∞	$-\infty$	∞


y
8
6
4
2
-2
x
-4
-2
2
4
(0,2)
(-4,0)
(1/2,0)