2 Functions

2.1 What is a Function?

Definition 2.1 *A* **function** *is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. Here the set A is called the* **domain** *of the function. We will also write*

$$f : A \to B$$

So f is an rule that takes an input x and produces an output $f(x)$. So if the input is 3, the output is $f(3)$ and so on.

Examples:

1. Let $f(x) = x^2$. Find $f(3), f(-2)$ and $f(\sqrt{5})$.

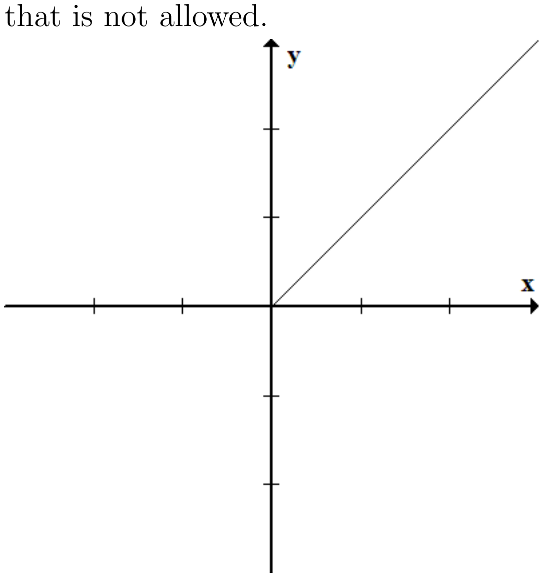
   solution: $f(3) = 3^2 = 9$, $f(-2) = (-2)^2 = 4$, $f(\sqrt{5}) = \sqrt{5}^2 = 5$.

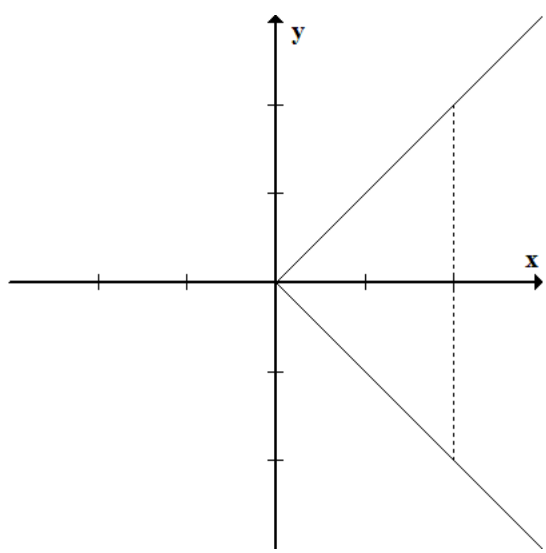
2. Let $f(x) = 3x^2 + x - 5$. Evaluate $f(-2)$ and $f(0)$.

   solution:

$$f(-2) = 3(-2)^2 + (-2) - 5 = 5$$
$$f(0) = 3(0)^2 + (0) - 5 = -5$$

3. There are two graphs below. The first is the graph of the function, while the second is not. The second is not because the same x-value gives two different y-values, and that is not allowed.





This is a function This is not a function

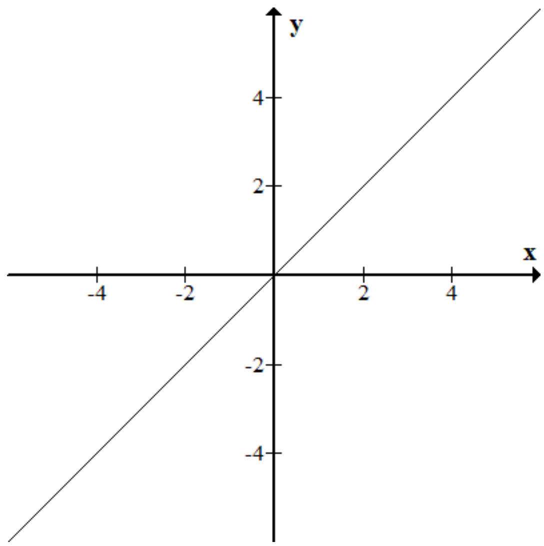
Definition 2.2 *Let A be the domain of $f(x)$. Then the* **range** *of f is the set of all possible values of $f(x)$ as x varies throughout the domain. That is*

$$\text{Range of } f = \{f(x) \mid x \in A\}$$

The range can also be thought of as **the set of attainable y values**.

Examples:

1. Let $f(x) = x$. Then every x value is allowed, so the domain is $\mathbb{R}$. From the graph, we see that every y value is attained, so the range is also $\mathbb{R}$.



2. Let $f(x) = x^2$. Again, every x value is allowed, so the domain is $\mathbb{R}$. This time though, f only outputs positive or 0 values. Hence the range is $f(x) \geq 0$ or, in interval notation, $[0, \infty)$.

2.2 Graphs of Functions

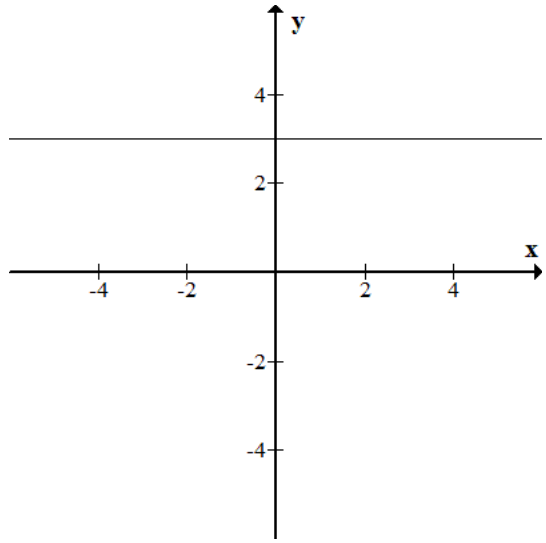
The graph of a function f is the set of all points (x, y) such that $y = f(x)$.

Examples: Sketch the graphs of the following functions

1. $f(x) = 3$

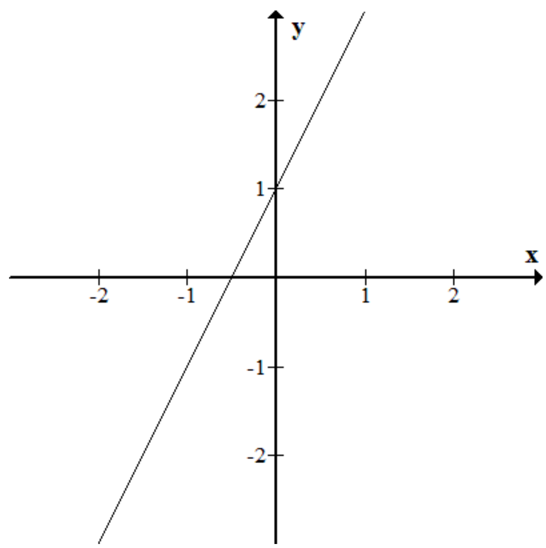
   solution:

x	$y = 3$	(x, y)
-2	3	$(-2, 3)$
-1	3	$(-1, 3)$
0	3	$(0, 3)$
1	3	$(1, 3)$
2	3	$(2, 3)$



2. $f(x) = 2x + 1$

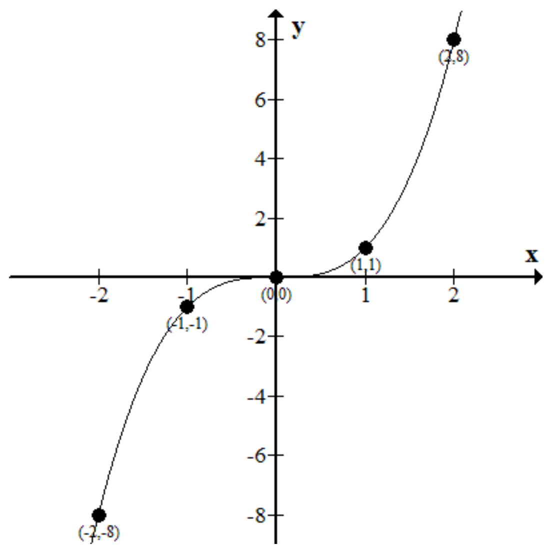
   <u>*solution:*</u> Here we don't need a table of values because we know this is the equation of a line. The slope is 2 while the y-intercept is 1.



3. $f(x) = x^3$

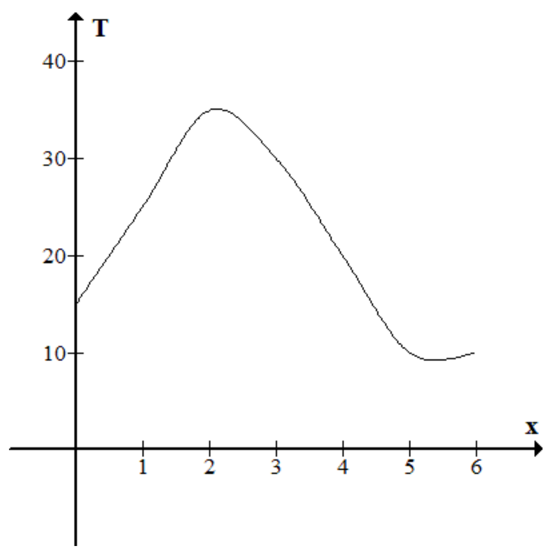
   <u>*solution:*</u>

x	$y = x^3$	(x, y)
-2	-8	$(-2, -8)$
-1	-1	$(-1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	8	$(2, 8)$



Getting Information from the Graph of a Function

Example: The function T graphed below gives the temperature between noon and 6pm at a certain weather station. The x values indicate hours from noon.



a) What are the temperatures at 1pm, 2pm and 5pm?

solution: The temperatures asked for correspond to the values $T(1), T(2)$ and $T(5)$. Reading off the graph, we see the temperatures are

$$T(1) = 25$$

$$T(2) = 35$$

$$T(5) = 10$$

b) When is it hotter, 2pm or 4pm?

solution: $T(2) = 35$ while $T(4) = 20$, so it is hotter at 2pm.

Finding the Domain and Range from a Graph

In the above example, we see that the allowed x values are between 0 and 6 including the endpoints, so the domain is $[0, 6]$. The range is the set of attained x-values which appear to be between 10 and 35. So the range is $[10, 35]$.

2.3 Increasing and Decreasing Functions; Average Rate of Change

Definition 2.3 *Let f be a function.*

- *f is* **increasing** *on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.*
- *f is* **decreasing** *on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.*

A function is increasing anywhere its graph is sloping upwards, and decreasing anywhere its graph is sloping downwards.

Average Rate of Change

Definition 2.4 *The* **average rate of change** *of the function $y = f(x)$ between $x = a$ and $x = b$ is*

$$\text{average rate of change } = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Example: Calculate the average rate of change for the function $f(x) = -4x^2 + 2x + 2$ between (a) $x = 0$ and $x = 2$ and (b) $x = 2$ and $x = 4$.

solution:

(a) $$\text{average rate of change} = \frac{f(2) - f(0)}{2 - 0} = \frac{-4(2)^2 + 2(2) + 2 - 2}{2} = \frac{-12}{2} = -6$$

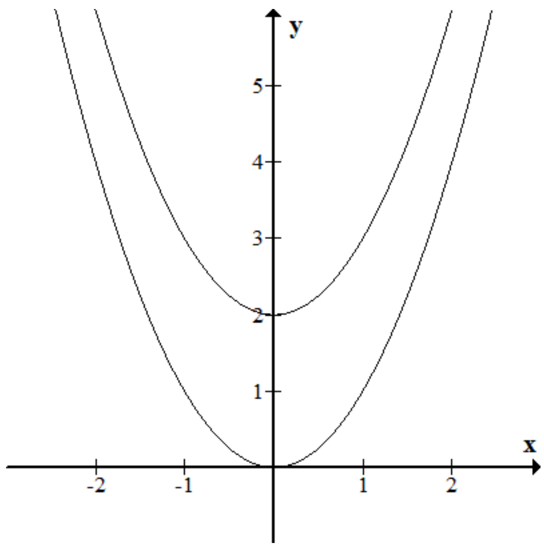
(b) $$\text{average rate of change} = \frac{f(4) - f(2)}{4 - 2} = \frac{(-4(4)^2 + 2(4) + 2) - (-4(2)^2 + 2(2) + 2)}{2} =$$

$$= \frac{-54 - (-10)}{2} = -22$$

2.4 Transformations of Functions

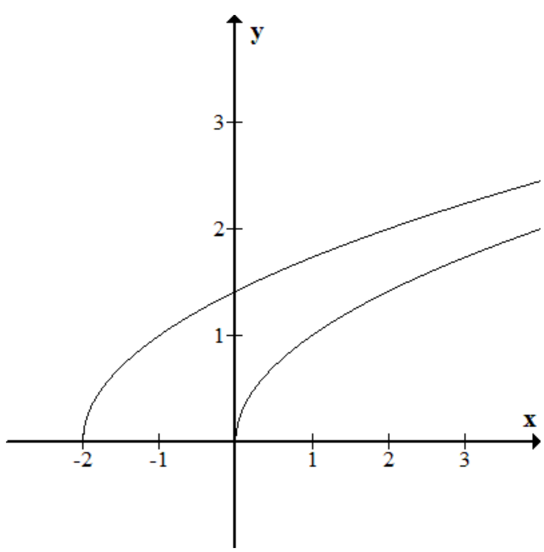
Vertical Shifting

The figure below shows a sketch of the graphs of $y = x^2$ and $y = x^2 + 2$. Here we obtain the graph of $y = x^2 + 2$ by shifting the graph of $y = x^2$ up by two units.



Horizontal Shifting

The figure below shows a sketch of the graphs of $y = \sqrt{x}$ and $y = \sqrt{x+2}$. Here we obtain the graph of $y = \sqrt{x+2}$ by shifting the graph of $y = \sqrt{x}$ **left** by two units.



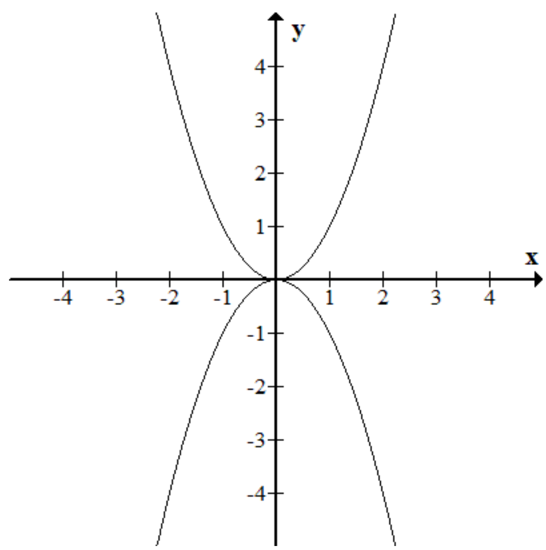
In General:

Suppose $c > 0$.

1. To graph $y = f(x) + c$, shift the graph $y = f(x)$ upwards c units.
2. To graph $y = f(x) - c$, shift the graph $y = f(x)$ downwards c units.
3. To graph $y = f(x + c)$, shift the graph $y = f(x)$ left c units.
4. To graph $y = f(x - c)$, shift the graph $y = f(x)$ right c units.

Reflecting Graphs

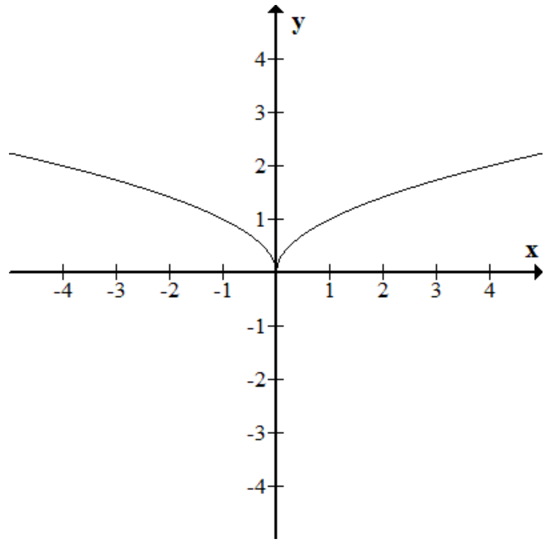
The figure below shows a sketch of the graphs $y = x^2$ and $y = -x^2$. Here, we can obtain the curve $y = -x^2$ by reflecting $y = x^2$ in the x-axis.



Likewise, the figure below shows a sketch of the graphs $y = \sqrt{x}$ and $y = \sqrt{-x}$. Here, we can obtain the curve $y = \sqrt{-x}$ by reflecting $y = \sqrt{x}$ in the y-axis.

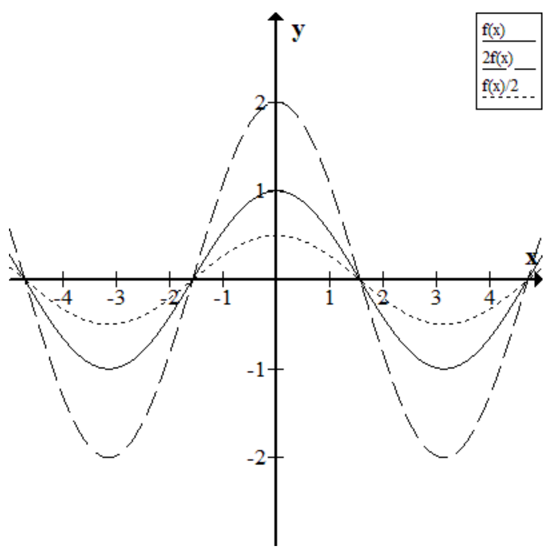
In General:

1. To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
2. To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



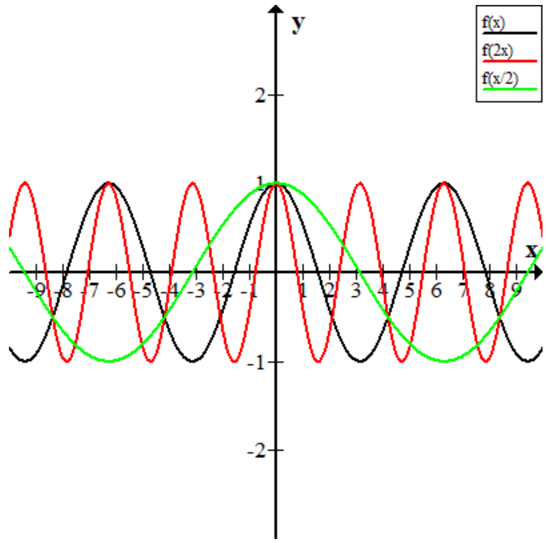
Vertical Stretching and Shrinking

The figure below shows the graphs of a function $f(x)$ along with $2f(x)$ in the dashed line and $\frac{1}{2}f(x)$ in the dotted line. Multiplying by 2 stretched the graph vertically while dividing by 2 shrunk it vertically.



Horizontal Stretching and Shrinking

The figure below shows the graphs of a function $f(x)$ along with $f(2x)$ in the red line and $f(\frac{1}{2}x)$ in the green line. Multiplying by 2 shrunk the graph horizontally while dividing by 2 stretched it horizontally.



In General:

To graph $y = cf(x)$:

- if $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.
- if $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.

To graph $y = f(cx)$:

- if $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of c.
- if $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of c.

Even and Odd Functions

Definition 2.5 *Let f be a function. We call f* **even** *if $f(-x) = f(x)$ for all x in the domain of f. We call f* **odd** *if $f(-x) = -f(x)$ for all x in the domain of f.*

Examples:

1. $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$
2. $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$

The graphs of even functions are symmetric about the y axis while the graphs of odd functions are antisymmetric about the y axis.

2.5 Quadratic Functions: Maxima and Minima

Graphing a Quadratic Function

A quadratic function is one of the form $f(x) = ax^2+bx+c$. To graph it, we must rearrange the terms.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a(x^2 + \frac{b}{a}x) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

If we let $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, then we get

$$f(x) = a(x-h)^2 + k$$

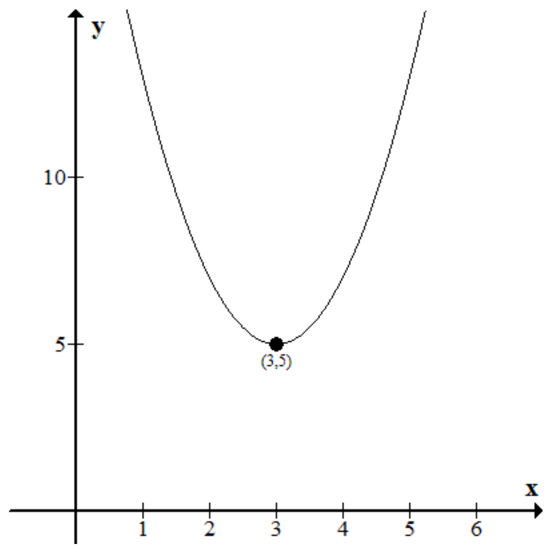
This is called **standard form** for the quadratic. From the standard form it is easy to see what the graph should look like. The graph of f is a parabola with vertex (h, k); the parabola opens upwards if $a > 0$ and downwards if $a < 0$.

Example: Sketch the graph of $f(x) = 2x^2 - 12x + 23$.

solution: Don't try to memorize the formula. Instead, try to complete the square each time. First we must factor out the 2 from each term containing an x, and then add and subtract a term that will make the thing inside the brackets a perfect square.

$$\begin{aligned} f(x) &= 2x^2 - 12x + 23 \\ &= 2(x^2 - 6x) + 23 \\ &= 2(x^2 - 6x + 9 - 9) + 23 \\ &= 2(x^2 - 6x + 9) - 18 + 23 \\ &= 2(x-3)^2 + 5 \end{aligned}$$

So $f(x)$ is a parabola with vertex at $(3, 5)$ opening up and stretched by a factor of 2.



Maximum and Minimum Values of Quadratic Functions

If a parabola opens up, then the minimum value for f occurs at $x = h$. If a parabola opens down, its maximum value occurs at $x = h$.

Example: In our example above, the parabola opens up, so its minimum value occurs at $x = 3$ (the minimum value is $f(3) = 5$).

2.7 Combining Functions

Rules: Let f be a function with domaind D_f and let D_g be a function with domain B. Then you can add, subtract, multiply, and divide the two functions wherever the domains overlap.

Operation	**Domain**
1. $(f+g)(x) = f(x) + g(x)$	$D_f \cap D_g$
2. $(f-g)(x) = f(x) - g(x)$	$D_f \cap D_g$
3. $(fg)(x) = f(x)g(x)$	$D_f \cap D_g$
4. $\left(\frac{f}{g}\right)$	$\{x \in D_f \cap D_g \mid g(x) \neq 0\}$

Example: Let $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$. Then $(f+g)(x) = \frac{1}{x-2} + \sqrt{x}$. Here, x cannot be negative or 2. Hence the domain is $\{x \mid x \geq 0, x \neq 2\}$, which is the intersection of the domain of f ($\{x \mid x \neq 2\}$) and the domain of g ($\{x \mid x \geq 0\}$).

On the other hand, $\left(\frac{f}{g}\right)(x) = \frac{1}{\sqrt{x}(x-2)}$. The domain of this function is the same as the above except we have to exclude the points where g is 0. Hence, the domain is $\{x \mid x > 0, x \neq 2\}$

Composition of Functions

An important way of combining functions is through composition.

Definition 2.6 *Given two functions f and g the* **composite function** $f \circ g$ *(also called the* **composition** *of f and g) is defined by*

$$f \circ g(x) = f(g(x))$$

So to do $f \circ g$, we do g first and then do f.

Examples:

1. Let $f(x) = x^2$ and $g(x) = x - 3$.

   (a) Find the functions $f \circ g$ and $g \circ f$ and their domains.
   solution:

   $$f \circ g(x) = f(g(x)) = f(x-3) = (x-3)^2$$
   $$g \circ f(x) = g(f(x)) = g(x^2) = x^2 - 3$$

   Both have domain equal to $\mathbb{R}$

   (b) Find $f \circ g(5)$ and $g \circ f(5)$
   solution:

   $$f \circ g(5) = (5-3)^2 = 4$$
   $$g \circ f(5) = 5^2 - 3 = 22$$

2. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$.

   (a) Find $f \circ g$ and its domain.
   solution:

   $$f \circ g(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = ((2-x)^{1/2})^{1/2} = (2-x)^{1/4} = \sqrt[4]{2-x}$$

   Fourth roots are only defined for nonnegative numbers, so we need $2-x \geq 0$, that is, $x \leq 2$. So the domain is $(-\infty, 2]$.

   (b) Find $g \circ f$ and its domain.
   solution:

   $$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}$$

   The inner root means we need $x \geq 0$. In addition, we need $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$. Squaring both sides gives is $x \leq 4$. Thus our domain is $[0, \infty) \cap (-\infty, 4] = [0, 4]$.

   (c) Find $f \circ f$ and its domain.
   solution:

   $$f \circ f(x) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

   Again, fourth roots are only defined for nonnegative numbers, so our domain is $[0, \infty)$

(d) Find $g \circ g$ and its domain.

<u>*solution:*</u>

$$g \circ g(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

For the inner root we need $x \leq 2$. For the outer root, we need

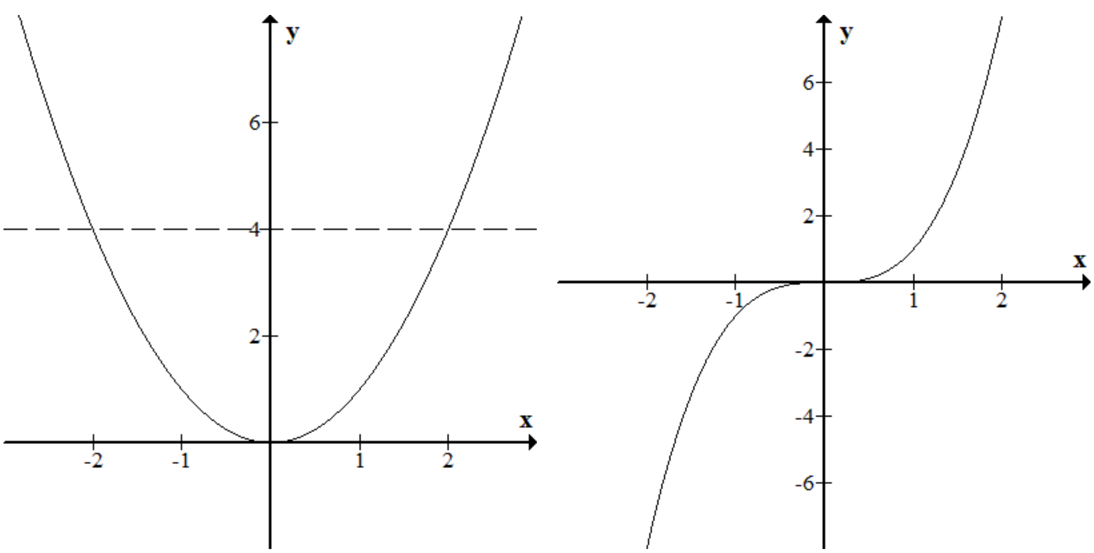
$$\begin{aligned} 2-\sqrt{2-x} &\geq 0 \qquad (1)\\ 2 &\geq \sqrt{2-x} \qquad (2)\\ 4 &\geq 2-x \qquad (3)\\ x &\geq 2-4=-2 \qquad (4) \end{aligned}$$

So our domain is $[-2, 2]$

2.8 One-to-one Functions and Their Inverse

Definition 2.7 *A function with domain A is called* **one-to-one** *if no two elements of A have the same image, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.*

The most common way to tell if a function is one-to-one is the **horizontal line test**. A function is one-to-one if no horizontal line intersects its graph more than once. The function $f(x) = x^2$ is not one-to-one because, for example, both 2 and -2 get sent by f to the same value (4). The function $g(x) = x^3$ is one-to-one because every number has a unique cube root.



x^2 fails the horizontal line test but x^3 does not.

Definition 2.8 *Let f be a one-to-one function with domain A and rance B. Then its* **inverse function**, *denoted f^{-1} has domain B and range A and is defined by*

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \text{ for any } y \in B$$

The way to find an inverse function is to take your equation $y = f(x)$, have x and y trade places, then solve for y

Examples:

1. Find the inverse of $f(x) = 3x + 2$

   solution: We are given $y = 3x + 2$, so we should have x and y switch places and then solve for y.

$$\begin{aligned} x &= 3y + 2 \\ x - 2 &= 3y \\ y &= \frac{x-2}{3} \end{aligned}$$

   So $f^{-1}(x) = \frac{x-2}{3}$

2. Find the inverse of $f(x) = \sqrt{x-2}$.

   solution: Once again we get x and y to switch places then solve

$$\begin{aligned} x &= \sqrt{y-2} \\ x^2 &= y - 2 \\ y &= x^2 + 2 \end{aligned}$$

   So $f^{-1}(x) = x^2 + 2$.